

2015

## From arrays to algebra

Lorraine Day

University of Notre Dame Australia, Lorraine.Day@nd.edu.au

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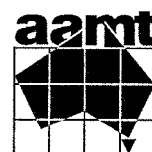


# mathematics: learn, lead, link



Proceedings of the 25th Biennial Conference of the  
Australian Association of Mathematics Teachers Inc.

Edited by N. Davis, K. Manuel & T. Spencer



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## PREFACE

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“Mathematics: Learn, Lead, Link” is an appropriate theme for the 25<sup>th</sup> Biennial Conference of the Australian Association of Mathematics Teachers. In a time of upheaval and change at every social scale, all representative bodies and their individual members must learn how to manage new and diverse challenges. That learning must be translated into transformational leadership that inspires and supports practitioners to make effective changes in their local spheres. I believe that this conference is a very important link in the network of change that many hope will lead to a future general populace that is more mathematically aware and competent.

Among the major themes that are very publicly connected to education are national security, financial literacy, environmental change, STEM, the apparent failure of our students in international rankings, national standards and the professionalism and quality of our teaching force, and the continuing marginalisation of Aboriginal Australians. It is true that schools have a fundamental role to play in dealing with social problems. However, calls from the public to solve this or that ill by delivery of additional curriculum are multitudinous and often seem simplistic, without genuine understanding of the complexities of education.

Fortunately, our conference speakers, as a community, with diverse interests, affiliations and experience, have the new ideas and the vital information that can support transformational learning and change. Coming together as a large and diverse group enables us to make links to fellow educators and to share knowledge and innovation. It is undoubtedly true that educators today must continue to learn and act upon their learning, for public good, as well as personal benefit. Every individual needs to understand more deeply the ways in which we can act to lead others and be an effective part of the network of change.

That which encourages me greatly in this endeavour is the tremendous support that educators give to each other, which I hope you experience in your workplace. In the context of this conference, I am grateful for the willing and generous help from many colleagues, most of whom I have never met, in carrying out my small role in the proceedings.

That which challenges me, is how we can best extend important ideas beyond our community of educators, to those who influence the thinking of other groups in society. I am conscious then, of a responsibility to make the most of the opportunity provided by the conference. I must think deeply about what influence I might have as an educator in my particular station; what learning is most essential to be an effective participant, and leader, in my community. It is obvious to me that I can make many

links to colleagues at the conference. I believe that from these links I could develop new understandings that may enable me to create meaningful connections with people, and exert a positive influence, however small, on society at large.

Neil Davis  
*Proceedings Lead Editor*

## **Review process**

Presentations at AAMT 2015 were selected in a variety of ways. Keynote and major presenters were invited to be part of the conference and to have papers published in these proceedings. A call was made for other presentations in the form of either a seminar or a workshop. Seminars and workshops were selected as suitable for the conference based on the presenters' submission of a formal abstract and further explanation of the proposed presentation.

Authors of seminar and workshop proposals approved for presentation at the conference were also invited to submit written papers to be included in these proceedings. These written papers were reviewed without any author identification (blind) by at least two reviewers. Reviewers were chosen by the editors to reflect a range of professional settings. Papers that passed the review process have been collected in the 'Papers' section of these proceedings.

The panel of people to whom papers were sent for review was extensive and the editors wish to thank them all:

Judy Anderson	Barry Kissane	Noemi Reynolds
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# FROM ARRAYS TO ALGEBRA

LORRAINE DAY

The University of Notre Dame Australia

Lorraine.Day@nd.edu.au

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By identifying the learning experiences that need to be developed in the primary school, and how secondary teachers can build on these experiences, ensures solid algebraic understanding and a smooth transition between primary and secondary mathematics. Using concrete materials, within a concrete–representational–abstract pedagogical approach, may be seen as one among many methods that contribute to the overall process of developing algebraic skills. Linking an area-based model to previous understandings involving numbers should assist the conceptual understanding of algebraic expansion and factorisation to ensure that students are not just fluent in algebraic manipulation but also understand how the processes work.

## Introduction

As a secondary trained teacher who now finds herself predominantly teaching primary pre-service teachers, an interest in the important transition between primary and secondary mathematics teaching has been developed. By looking at the big picture, it is possible to identify the learning experiences that need to be developed in the primary years and how secondary teachers can build on these experiences to ensure a smooth transition and solid mathematical outcomes.

One of the big ideas to set children up for algebra in secondary schools is developing multiplicative thinking, and an important part of developing multiplicative thinking is developing an understanding of the importance of arrays as multiplicative models. As early as Year 2 the *Australian Curriculum: Mathematics* states that students should “recognise and represent multiplication as repeated addition, groups and arrays” (ACARA, 2014). In particular, arrays and regions assist in supporting the shift from additive thinking (‘groups of’ model) to multiplicative thinking (‘factor-factor-product’ model; Siemon, 2013), and eventually to proportional and algebraic thinking.

By laying appropriate groundwork, primary teachers can set children up for not only a deeper and more robust understanding of multiplication and division, but also mathematical success in the secondary years.

## A developmental approach

When students are asked to use blocks, tiles or counters to make four groups of three they will often make a model that resembles Figure 1. This representation is correct, but if the aim is to move students from additive to multiplicative thinking, may not be the most efficacious. The model used in Figure 1 can encourage students to think about multiplication only as repeated addition, whereas an array or region model (Figure 2) not only demonstrates the strategy of repeated addition it also encourages other understandings such as the relationship between factors and multiples and the link of multiplication with area.

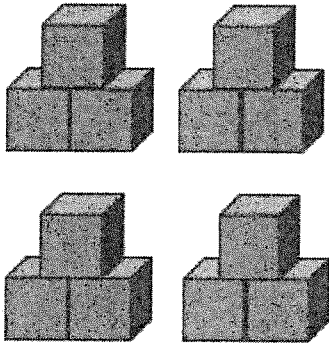


Figure 1. Four groups of three.

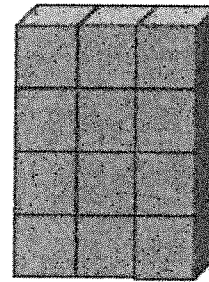
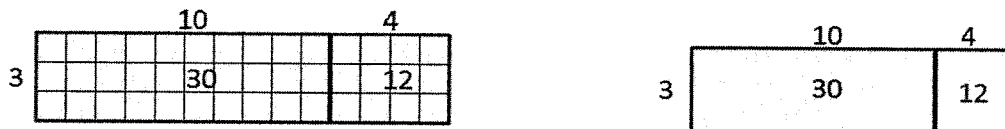


Figure 2. Array: four groups of three.

Array models can be extended into other multiplication situations. Representing a one-digit by two-digit multiplication as an array provides a visualisation of both the magnitude of a number and how it can be partitioned to promote understanding. It also demonstrates the link between multiplication, the distributive property and area (see Figure 3).

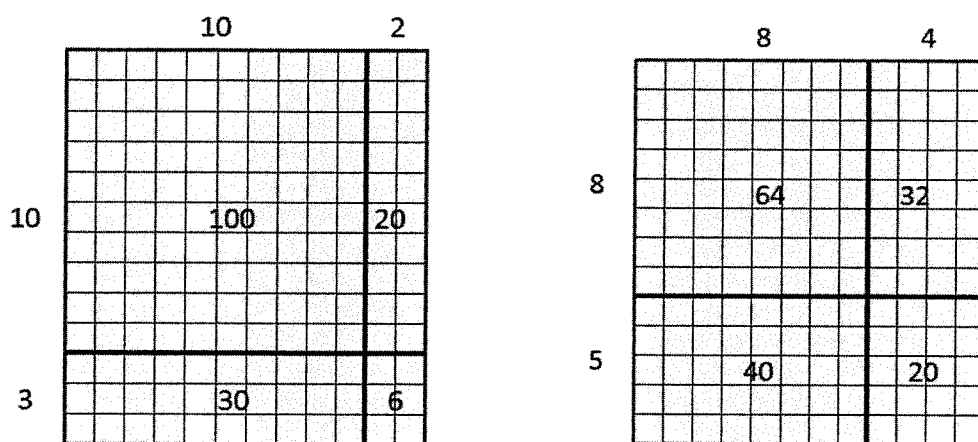


$$3 \times 14 = 3 \times 10 + 3 \times 4 = 30 + 12 = 42$$

Figure 3. Array representations of  $3 \times 14$ .

This idea can then be extended to two-digit by two-digit multiplication, or any-digit-by-any-digit multiplication. Students who do not use an array or region method to visualise multiplication and who have a limited understanding of the distributive property often think that  $13 \times 12$  can be calculated by  $10 \times 10 + 3 \times 2$ , which is incorrect. Using the array model and identifying the area of the associated regions allows students to identify why this is not the case (see Figure 4).





$$\begin{aligned}
 13 \times 12 &= 10 \times 10 + 3 \times 10 + 10 \times 2 + 3 \times 2 \\
 &= 100 + 30 + 20 + 6 \\
 &= 156
 \end{aligned}$$

$$\begin{aligned}
 13 \times 12 &= 8 \times 8 + 5 \times 8 + 8 \times 4 + 5 \times 4 \\
 &= 64 + 40 + 32 + 20 \\
 &= 156
 \end{aligned}$$

Figure 4. Array representations of  $13 \times 12$ .

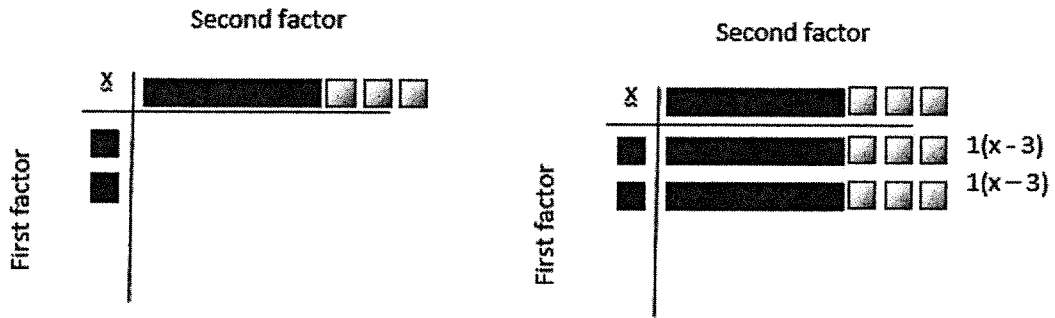
Building on previous understandings is important. When students are able to build on prior knowledge by making connections to previous understandings, their learning will be more meaningful. For example, just as students should see that the process of multiplying by an any-digit number is just an extension of the process for multiplying by a one-digit number, they should understand that the process is the same no matter what the multiplier (Reys et al., 2012). This, in turn, leads to the process of generalising and allows students to apply this prior knowledge to algebraic representations, forging the links of factors and the distributive property from number and applying them to algebra.

This development of the understanding then continues into the secondary classroom. The use of a concrete, visual, area-based model, such as algebra tiles allows this seamless transition. Algebra tiles are a model that can facilitate this. Although algebra tiles are versatile, they are not intended alone to constitute an entire course in algebraic manipulation but instead are anticipated to be one of several ways in which conceptual understanding can be developed. To use algebra tiles, students need only to understand the properties of the additive identity ( $a + 0 = a$ ) and the additive inverse ( $a + (-a) = 0$ ).

Building on the array model for multiplication of any-digit numbers, linear and quadratic expansions and factorisation may be modelled with algebra tiles. They are also useful for investigating integer arithmetic, but that is beyond the scope of this paper.

By looking at  $2(x - 3)$  as two lots of  $(x - 3)$  a model can be constructed using algebra tiles (see Figure 5). The use of the terms ‘factors’ and ‘multiple’ which were introduced when students were looking at arrays in the Number strand in the primary years are illustrated in a meaningful way.

$$2(x - 3)$$

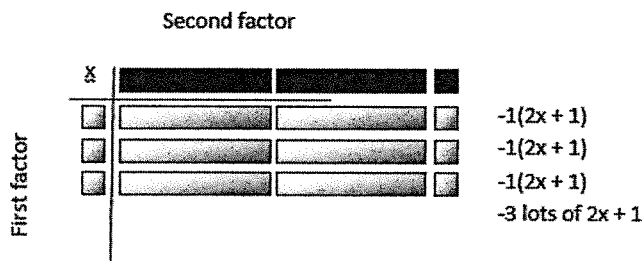


The answer is  $2x - 6$ , which is a rectangle with length  $x - 3$  and width 2.

Figure 5. Expanding  $2(x - 3)$ .

In the same way, multiplying by a negative number may also be modelled (see Figure 6).

$$-3(2x + 1)$$



The answer is  $-6x - 3$  which is a rectangle of length  $2x + 1$  and width  $-3$ .

Figure 6. Expanding  $-3(2x + 1)$ .

Linear factorisation builds on previous understandings of the array models of prime and composite numbers that have been developed in the primary years. Students can use algebra tiles to consolidate that prime numbers (and the number 1) can only be represented by one array, whereas composite numbers may be represented by at least two different arrays (See Figures 7 and 8).



Figure 7. Seven is prime.

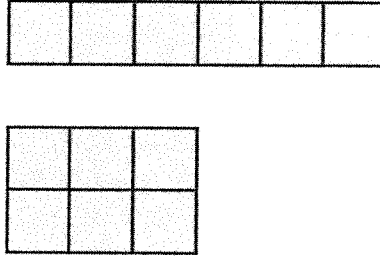


Figure 8. Six is composite.

In the same manner,  $2x + 5$  can only be represented by one array, so it has no integral factors whereas  $2x + 6$  may be represented by more than one array which means it has integral factors (see Figure 9). Naturally, if there is a positive integral factor, then there is also a negative integral factor.

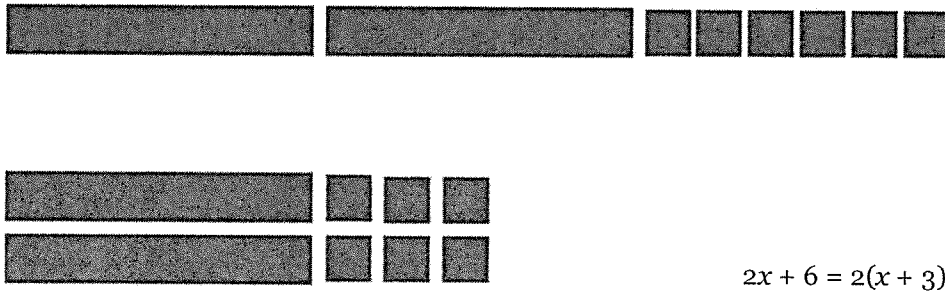


Figure 9.  $2x + 6$  may be represented by two arrays, so has an integral factor.

Quadratic expansions may also be easily represented using algebra tiles. The area representation parallels the formal expansion method for number using the distributive property. The multipliers (factors) may be set up on the frame (see Figure 10);  $(x + 2)(x - 3)$  means all of  $(x + 2)$  is to be multiplied by all of  $(x - 3)$ .

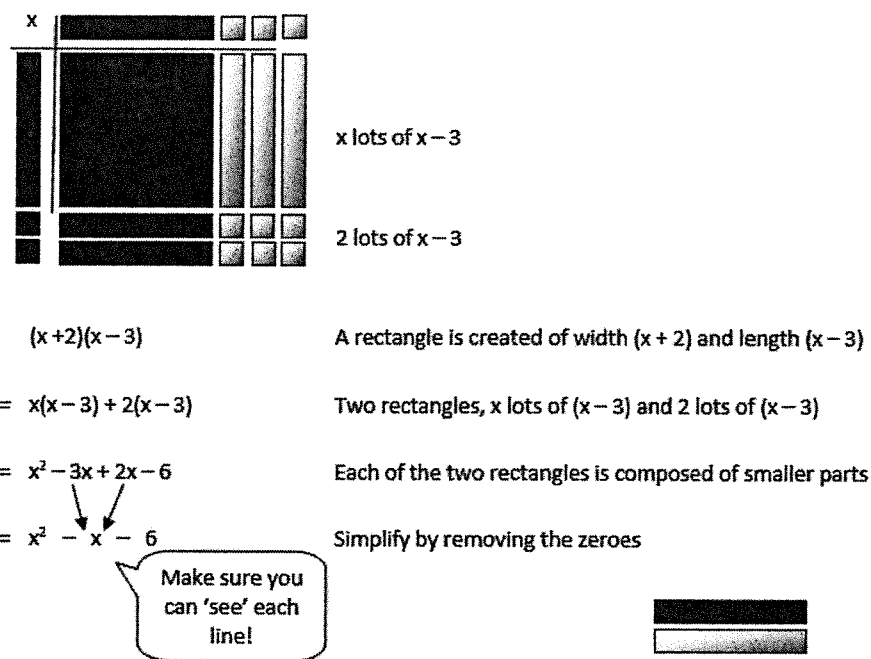


Figure 10. Expanding  $(x+2)(x-3)$ .

Each line of formal mathematical notation is accompanied by a concrete representation, which allows students to 'see' the meaning of each line. Other investigations of perfect squares, difference of perfect squares and examples of other coefficients of  $x^2$  can also be explored. For a detailed description of how to factorise quadratics using algebra tiles see Day (2014).

The solution of linear equations may also be modelled using algebra tiles (See Figure 11), especially if students in their primary school years have had a good grounding in the equals sign representing balance. Once again, formal notation can be developed alongside the concrete model.

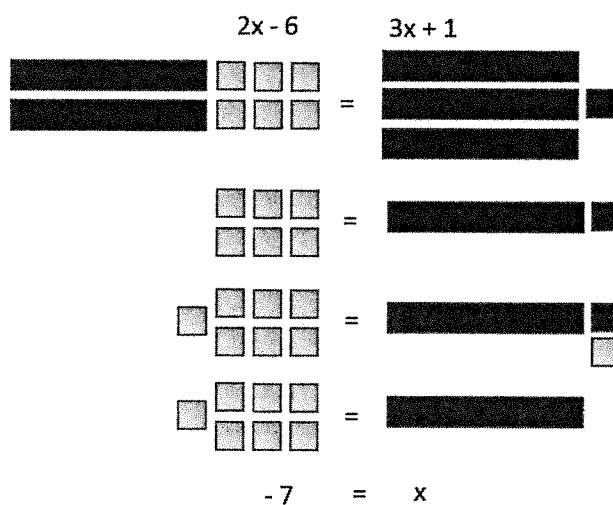


Figure 11. Solution of linear equation  $2x - 6 = 3x + 1$ .

There are some limitations of the algebra tiles model. To represent a variable with concrete materials is one such obvious limitation. To endeavour to overcome this, it is important that the length attributed to the variable is not a multiple of the length attributed to a unit piece. In this way students are less likely to form a visual impression of a numerical value for the variable. Although there are some limitations of the model, the advantages of having a concrete, visual, area-based model far outweigh the limitations (Lovitt, personal communication, June 26, 2013).

## The Australian Curriculum: Mathematics

The following are content descriptors from the *Australian Curriculum: Mathematics* (ACARA, 2013) that can be directly addressed by the use of algebra tiles.

- Introduce the concept of variables as a way of representing numbers using letters (ACMNA175)
- Create algebraic expressions and evaluate them by substituting a given value for each variable (ACMNA176)
- Extend and apply the laws and properties of arithmetic to algebraic terms and expressions (ACMNA177)
- Solve simple linear equations (ACMNA179)
- Extend and apply the distributive law to the expansion of algebraic expressions (ACMNA190)
- Factorise algebraic expressions by identifying numerical factors (ACMNA191)
- Solve linear equations using algebraic and graphical techniques. Verify solutions by substitution (ACMNA194)
- Apply the distributive law to the expansion of algebraic expressions, including binomials, and collect like terms where appropriate (ACMNA213)
- Factorise algebraic expressions by taking out a common algebraic factor (ACMNA230)
- Expand binomial products and factorise monic quadratic expressions using a variety of strategies (ACMNA233)
- Solve problems involving linear equations, including those derived from formulas (ACMNA235)

With the Proficiency Strands of Understanding, Fluency, Reasoning and Problem Solving as the power driving the content of the *Australian Curriculum: Mathematics*, there is increased motivation for teachers to ensure that students are not just procedurally fluent in algebraic manipulation but also understand how the processes work and how to reason mathematically. By linking the area-based model to previous understandings involving number, secondary teachers can build on the work done in primary schools and assist students to gain an understanding of the concepts underpinning algebraic manipulation. Having students work with concrete materials provides an ideal forum for mathematical conversations which assist students to clarify their own ideas and share ideas with others. This leads to explanation, justification and communication, all of which contribute to mathematical reasoning.

## Acknowledgment

Much of the work on algebra tiles is adapted with permission from Lovitt, Marriott and Swan (1979). In the words of Charles Lovitt: “The tiles are a model, a concrete, visual, area-based model, and all models have limitations ... but it is still one of the most powerful models I have found” (Lovitt, personal communication, June 26, 2013).

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