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ORIGINAL ARTICLE

Perceptions on proof and the teaching of proof: a comparison across preservice secondary teachers in Australia, USA and Korea

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Abstract
Despite the recognised importance of mathematical proof in secondary education, there is a limited but growing body of literature indicating how preservice secondary mathematics teachers (PSMTs) view proof and the teaching of proof. The purpose of this survey research was to investigate how PSMTs in Australia, the USA and Korea perceive of proof in the context of secondary mathematics teaching and learning. PSMTs were able to outline various mathematical and pedagogical aspects of proof, including purposes, characteristics, reasons for teaching and imposed constraints. In addition, PSMTs attended to differing, though overlapping, features of proof when asked to determine the extent to which proposed arguments constituted proofs or to decide which arguments they might present to students.

Keywords Reasoning and proof · Preservice teacher education · Mathematical knowledge for teaching · High school education · Secondary school education

Introduction

The centrality of proof to the discipline of mathematics is indisputable. However, the extent to which proof plays a significant role in the teaching and learning of

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mathematics across the grades is subject to variation and debate (Hanna and de Villiers 2008; Reid and Knipping 2010; Stylianou et al. 2009). Proof engages students in authentic mathematical practices—revealing the axiomatic structure of the discipline and the infallible nature of mathematical truths (Zaslavsky et al. 2012). More importantly, constructing mathematical arguments to convince oneself or others of statement's truth (or falsehood) provides opportunities for students to deepen their understanding of underlying mathematical ideas.

However, research has revealed that secondary mathematics teachers often hold a limited view on the purpose of proof instruction and its appropriateness for all students (Bergqvist 2005; Knuth 2002a; Kotelawala 2016). Views of proof as a formalistic mechanism are magnified for prospective teachers (Boyle et al. 2015; Varghese 2009) whose most recent experiences with proof are often in the context of upper-level mathematics courses. Without a more complete understanding of the nature of proof and its role in learning mathematics, it is unlikely that beginning teachers will be equipped to enact proof instruction for all students. To assist in this regard, we offer an analysis of preservice secondary mathematics teachers' perceptions of proof and their validations of proposed proofs. Our study complements previous international comparative research on teacher knowledge of proof (e.g. Schwarz et al. 2008) to strengthen the literature base and to provide direction for future research and practice in proof education.

The aim of this research was to (1) investigate how preservice secondary mathematics teachers (PSMTs) in the USA, Australia and South Korea perceived the purpose of proof in the context of secondary mathematics teaching and learning, and (2) determine what characteristics of proof PSMTs centralise in their evaluations of proofs. In the following section, we first provide our overarching perspective on teacher knowledge of proof before reviewing the literature pertaining to each of these aims, teachers' perceptions of the role of proof and how teachers evaluate student proofs.

Perspectives on mathematical knowledge for teaching proof

In concert with others (e.g. Baumert et al. 2010; Hill et al. 2008), we contend that teachers' instructional practices are constrained by a complex constellation of teacher knowledge, skills and dispositions. In our case, this includes mathematical content knowledge such as knowing what constitutes a proof, specialised knowledge of proof connected to the work of teaching (i.e. ability to recognise alternate forms of proof and evaluate a range of arguments) and views of the purpose of proof in school mathematics. Our view of mathematical knowledge for teaching proof is grounded in situative perspectives on teacher learning (Adler and Davis 2006; Putnam and Borko 2000) that take into account the various contexts and tasks through which knowledge is both acquired and deployed.

For example, preservice teachers encounter proof, and hence develop knowledge, skills and dispositions toward proof and teaching proof, in multiple settings that include their own lower and secondary schooling, university mathematics courses and methods courses within teacher education departments. The value of a situative perspective (i.e. attending to varied contexts and knowledge domains) is evidenced in studies demonstrating how teachers' conceptions of proof within the discipline of mathematics are sometimes at odds with their instructional practices and views on the role of proof in mathematics learning (e.g. Knuth 2002b; Kotelawala 2009; Peressini et al. 2004; Tsamir et al. 2009). This perspective underlies a number of frameworks that have been developed to delineate mathematical knowledge that supports the work of teaching proof (e.g. Lesseig 2016; Steele and Rogers 2012; Stylianides and Ball 2008). As described in the methods section below, Lesseig's (2016) Mathematical Knowledge for Teaching Proof (MKT for Proof) framework guided parts of our analysis. This framework (Table 1) provided an initial lens through which to analyse our data and grounded our subsequent interpretations of PSMTs' conceptions of proof and their proof evaluations.[1]

Teachers' perceptions of proof and the role of proof

In order to recognise and promote proof as a sense-making activity applicable to students at all grade levels, teachers need a comprehensive understanding of the role of proof (de Villiers 1990; Hanna 1990; Hanna 2000; Knuth 2002b). De Villiers (1990) described five distinct, but related roles proof serves in mathematics: verification, explanation, systematisation, discovery and communication. Greater attention to the explanatory, discovery and communicative roles of proof, he argued, could potentially make proof a more meaningful activity in mathematics classrooms. Research suggests that teachers differentially embrace these roles. Although teachers purportedly value proof as a means to build understanding (Dickerson and Doerr 2014; Knuth 2002a), in practice, teachers more often promote proof as a tool to verify results or to provide procedural accounts (Bieda 2010; Kotelawala 2016; Knuth 2002b).

For example, secondary teachers in Dickerson and Doerr's (2014) study stated that the two most important roles of proof were to build mathematical understanding and to develop general thinking skills relevant outside of mathematics (e.g. logical thinking and metacognitive skills). However, less experienced teachers were more apt to take a formalistic view of proof and place greater value on explicit logic and precise language in writing proofs (Dickerson and Doerr 2014). Kotelawala's (2016) study revealed that teachers who had developed personal anxiety or struggled with mathematics, as well as those with more mathematics coursework, were less likely to prioritise learning proof over mathematical procedures in their classroom. Teachers who had played an active role in proving within their mathematics education courses, however, were more likely to prioritise proving.

Teachers' evaluations of proof

Recent studies have investigated differences in how expert and novice mathematicians validate proofs (e.g. Inglis and Alcock 2012; Weber 2008) as well as the factors various groups attend to when assessing student-generated proofs (e.g. Bleiler et al. 2014; Ko and Knuth 2013; Miller et al. 2017; Moore 2016). Several findings across these studies are relevant to our understanding of how PSMTs evaluate proof. First, even mathematicians vary in their judgments of a proof's validity, indicating that deciding what makes

[1] See Lesseig (2011, 2016) for extensive literature review and empirical study that led to the development of the MKT for Proof framework.

Table 1 Mathematical Knowledge for Teaching Proof framework

Common content knowledge for proof	Specialised content knowledge for proof
Ability to construct valid proof • Understand and use stated assumptions, definitions & previously established results • Build a logical progression of statements • Analyse situations by cases • Use counterexamples	Explicit understanding of proof components • Accepted statements ° Range of useful definitions or theorems ° Role of language and defined terms • Modes of representation Variety of visual and symbolic methods to provide a general argument
Essential proof understandings • A theorem has no exceptions • A proof must be general • Proof is based on previously established truths • The validity of a proof depends on its logic structure	• Modes of argumentation °Recognise which methods (e.g. proof by exhaustion, counter-example) are sufficient or efficient °Identify characteristics of empirical and deductive arguments, including generic examples
Functions of proof • To establish the validity of a statement • To communicate math reasoning	Additional functions of proof • To provide insight into why the statement is true • To build mathematical understanding

an argument a proof is not necessarily based on objective, agreed-upon criteria (Inglis and Alcock 2012; Inglis et al. 2013).

Second, when judging the validity of a proof, novice mathematicians are more likely to focus on surface features, whereas experts first read proofs holistically to attend to the logical structure of the argument. Research indicates that secondary teachers also tend to focus on the format rather than the substance of a proof and have difficulty evaluating pictorial or verbal arguments (Dickerson and Doerr 2014; Tabach et al. 2010). Dickerson and Doerr's (2014) research suggested that teachers' ability to accept and promote informal modes of proof representation (e.g. visual and concrete models) develops over time, with more veteran teachers less likely to impose strict standards for mathematical language and format. This claim aligns with Bleiler et al. (2014) study wherein preservice teachers were more apt to identify errors in arguments presented in verbal versus symbolic form, even when the same logic error was present.

Finally, when evaluating student proofs, mathematicians looked for more than just logical correctness, rewarding proofs that were presented clearly and demonstrated understanding (Miller et al. 2017; Moore 2016). Mathematicians differed in the extent to which they focused on mathematical notation and precise language; however, all agreed that these characteristics bore little weight when assigning points. In short, mathematics professors were most interested in the quality of student reasoning and whether students understood *why* a claim was true; thus, they were willing to forgive imprecision.

Proof education in Australia, USA and South Korea

Proof in the Australian curriculum The *Australian Curriculum: Mathematics* (ACM) has been implemented in primary and secondary schools within each Australian state and territory since 2016. The ACM is organised according to three strands (Number and Algebra, Measurement and Geometry, Statistics and Probability), and the content descriptions of each strand are informed by four mathematical proficiencies (Understanding, Fluency, Problem Solving, and Reasoning). For upper primary and middle secondary education (Years 6–10), the Reasoning proficiency uses terminology related directly to proof ("formulate proofs", "prove and apply", "explain", "justify", "develop"). In the content descriptions for Years 8 and 9, reference is made to proof by way of congruence and similarity in shapes (e.g. triangles, quadrilaterals), and for the sine, cosine and tangent ratios in right-angled triangles. In Year 10, direct reference is made to formulating proofs involving congruent triangles and angle properties, as well as to similar triangles and other planar shapes. In addition, the content descriptions for the Year 10 Advanced course outline that students are required to prove and apply angle and chord properties of circles.

In senior secondary school (Years 11 and 12), proof is mentioned only in three courses: *General Mathematics*, *Mathematical Methods*, and *Specialist Mathematics*. Proof is mentioned in the *General Mathematics* syllabus where students deduce rules for the *n*th terms for both arithmetic and geometric sequences (ACARA 2018a). In *Mathematical Methods*, students are required to establish and use the formula for the sum of the first *n* terms of arithmetic and geometric sequences. There are also opportunities for students to prove and apply angle formulas and trigonometric identities using formal and informal methods including geometric constructions (ACARA 2018b). *Specialist Mathematics* students look at the nature of proof in depth with close attention paid to circle properties and theorems, and to geometric proofs using vectors. Proof is revisited in the *Specialist Mathematics* course when students work with proofs involving rational and irrational numbers as well as established proof procedures of contradiction and mathematical induction (ACARA 2018c).

Proof in the United States curriculum The *Common Core State Standards for Mathematics* (CCSSM) were initiated in 2010 in an attempt to focus and unify the mathematics curriculum in the USA (CCSSI 2010). Although not all states have adopted these standards, they are illustrative of the universal effort to improve mathematics achievement and college readiness of US students. The CCSSM are comprised of both content standards and a set of eight overarching Standards for Mathematical Practice (SMPs). Similar to the Mathematical proficiencies within the Australian Curriculum, the SMPs describe mathematical habits of mind—ways of reasoning, communicating and using mathematical representations and tools—that students are expected to exhibit across all content strands and grade levels.

Specifically, the third SMP states that all students should be able to "construct viable arguments and critique the reasoning of others" (CCSSI 2010, p. 6). This practice includes understanding and using assumptions, constructing arguments, and making and exploring conjectures in a logical progression. Although the word "proof" is not used, the principles behind justification, argumentation and logical reasoning all parallel the basis of constructing viable proofs. Younger students are expected to engage in this practice by using drawings, concrete objects or actions, whereas older students are expected to compare multiple arguments, recognise correct logical reasoning and understand when reasoning is flawed.

For grades K-8 (Year 0 through Year 8), the content standards define what students are expected to know across multiple strands (e.g. Number and Operations, Measurement and Data, Geometry) depending on their year in school. In Years 9–12, the content standards are aimed at specific classes (e.g. Algebra, Geometry, Statistics and Probability) in order to account for differentiation in the types and levels of classes that US students take. Within the content standards, the word "proof" only appears in high school Geometry. However, the expectation that students will explain their reasoning and justify solutions or conclusions appears in multiple grade levels. At the elementary level, this is seen with regard to using multiplication to determine the area of a rectangle, decomposing fractions using visual models and developing formulas for areas of geometric figures. Later, students are expected to justify solution methods in high school Algebra, and to make inferences, justify conclusions and evaluate data-based reports in high school Statistics and Probability.

Proof in the Korean curriculum The current *Korean Mathematics Curriculum* (KMC; Ministry of Education 2015) has been implemented since 2015. The KMC is determined at the national level and has had significant influence on mathematics education in elementary (Years 1–6), junior high (Years 7–9) and senior high school (Years 10–12). In the current KMC, the word proof first appears in the standards for Year 10 in the Number and Operation area wherein students are expected to understand proposition, converse of proposition and contraposition, methods of proof using contraposition and prove simple absolute inequality. These standards represent a retreat of proof education compared to the previous KMC. Until 2008, the standards for proof education first appeared in Years 8–9 of junior high school in the Geometry area, where students were expected to understand the meaning of proposition and proof, prove the properties of triangles and quadrilaterals and prove the Pythagorean Theorem. Such standards have been changed in the current KMC to *understand and explain* the properties of triangles and quadrilaterals, and *understand and explain* the Pythagorean Theorem. These revised standards run counter to the status of proof education in many other countries. Notably, the mathematics curriculum of China and Japan deals with proofs of theorems on triangles and quadrilaterals and the Pythagorean theorem in Years 8–9 (Ministry of Education, Culture, Sports, Science and Technology of Japan 2008; Ministry of Education of the People's Republic of China 2011).

In the background of this weakening of proof education in Korea, there were concerns that many junior high school students have difficulties in learning proof and abandon mathematics as a result. Some teacher and civic groups in Korea have consistently argued that it is necessary to weaken the proof education in junior high school because students lose interest in mathematics learning due to over-teaching of proofs. Reflecting the issues, the Ministry of Education of Korea decided to change the word "prove" to "understand and explain" in Year 8–9 standards since 2009.

Summary of proof education in Australia, USA and South Korea An analysis of proof education across Australia, the USA and South Korea reveals several points of commonality and dissonance. To commence, proof education is similar in these countries curriculum documents inasmuch as the term "proof" is used for junior secondary students (Years 7–9) in Geometry (e.g. properties of triangles), although this particular term has not been used in South Korea since 2009. Second, concerning the explicit usage of "proof" in curriculum documents, the US and South Korea documents both incorporate a number of different terms to describe mathematical concepts and activities directly involving proof. Third, and in contrast to the other two countries, the ACM for senior secondary school appears to contain the most explicit language for using proof in mathematical concepts, including specific topics in mathematical proof and reasoning.

Methods

Participants

The purpose of this pilot study was to investigate how PSMTs in Australia, the USA and Korea perceived the purpose of proof and features to which they attended when evaluating proofs in the context of secondary mathematics teaching and learning. Data for this paper come from a survey instrument completed by 34 students enrolled in mathematics teacher education programs in the USA (11), Australia (11) or Korea (12).

The 11 PSMTs from Australia were all enrolled at the university at which the second author teaches. At this university, PSMTs have three options for secondary mathematics teacher preparation. One option is for students to receive a Bachelor's Degree in Secondary Education (BEd), where PSMTs graduate with 32 weeks of practicum experience spread out over their four years in the program. Students in this program take eight mathematics content courses, one of which prominently features proof. The second option, which is to receive a Master of Teaching in Secondary Education (MTeach), is a two-year program with 20 weeks of practicum experience. The third option, which is to receive a Graduate Diploma of Secondary Education (GDE), includes students who enter this one-year program with an undergraduate degree in a related content field. Students in both graduate level programs are required to have taken eight mathematics content courses (for a major in mathematics education), or four mathematics content courses (for a minor). For all three options, students are required to complete one secondary mathematics methods course, which focuses on best practice approaches for curriculum planning, pedagogy and assessment. It was during this course that second year BEd students, first year MTeach students and GDE students were invited to complete the survey instrument.

The 11 PSMTs from the US were split among five different universities. Students earning a Bachelor's Degree at these universities have the option to major in mathematics education or in mathematics with either a concentration in teaching or additional licensure. Alternatively, students may pursue a Master's Degree in Education with a teaching option. Given the variety of programs and available pathways, the number of practicum hours and student teaching experiences ranges from none to a full year. All PSMTs in our study had taken at least three college level mathematics courses, with the majority of PSMTs (9) having taken 10 or more college level mathematics courses. Nine of the 11 reported having taken a course focusing on proof, citing courses such as Abstract Algebra, Analysis, Discrete Mathematics and Geometry. Although we are unsure as to the exact number of Education courses taken by these students, we know that four of these students were in their first year of the program, six of these students were in their second year of the program and one student was in the third year of the program. Based on this information as well as the required courses and planned trajectories from each respective program, it is reasonable to assume that six of these participants have had at least one to two education courses, with the possibility of the third year student having had four education courses. Here, we include teaching, curriculum and instruction, education, and education psychology courses.

The 12 PSMTs from Korea were from three different universities with four participants from each university. Each participant attended the Department of Mathematics Education within the College of Education and took the program for secondary mathematics teacher preparation, a Bachelor's Degree in Secondary Mathematics Education. Among 12 Korean PSMTs, six were in their second year and the other six were in their third year of the program. All of them reported having taken 10 or more university level mathematics courses focusing on proof. The second year students cited Advanced Calculus, Linear Algebra, Number Theory, Geometry, Set Theory and Discrete Mathematics. The third year students cited Modern Algebra, Topology, Complex Analysis and Differential Geometry including the above mathematics courses. In addition to, all Korean PSMTs took at least two courses on mathematics education, which focus on mathematics curriculum, mathematics education psychology, mathematics teaching method, mathematics learning theory and mathematics assessment. PSMTs graduate with four full weeks of practicum experience spread out in the fourth year in the program.

Survey instrument construction

The survey instrument was developed based on proof literature detailing essential elements of proof and proof instruction (e.g. Ellis et al. 2012; NCTM 2009) and drew heavily on Knuth's (2002a, b) prior study with practicing teachers. After answering basic demographic questions (e.g. year in respective teacher preparation program, number of mathematics courses taken), participants responded to questions in three distinct sections of the survey instrument. We describe each of these three parts next. The survey instrument questions are included in Table 7 in Appendix.

Part I of the survey instrument, comprised of open components and two Likert-scale items, focused on PSMTs' conceptions of proof in mathematics and in teaching mathematics. Two questions elicited PSMTs' views on the purposes of proof in mathematics and what makes an argument proof, which were related to the "Big Ideas and Essential Understandings" of proof proposed by Ellis et al. (2012). Big ideas 2 and 3 describe proof as a specific argument comprised of a sequence of deductive, logical statements, which demonstrates the truth of a mathematical statement for all possible cases. Big idea 5 refers to the many roles of proof, which closely align with those proposed by de Villiers (1990). These purposes include to verify the truth or falsehood of a statement, to provide insight or impetus to develop a new theory and to create a structure for communicating and using precise mathematical language. The remaining four questions asked PSMTs to rate the necessity of proof teaching and degree to which they anticipate planning lessons on proof, and to describe both reasons for teaching proof and constraints on teaching proof. These final questions, targeting secondary PSMTs' conceptions of proof in the context of school mathematics, were developed based on the results of Knuth's (2002b) study with practicing teachers (see Table 7 in Appendix).

Part II of the survey instrument focused on PSMTs' evaluation of arguments, using closed items (Yes/No this argument is a proof) with an open component to explain their response. PSMTs were given 10 student-generated arguments across five different statements and asked to determine whether the arguments constituted proof and explain why they were (or were not) proofs. Evaluating and making sense of others' arguments is one of the essential understandings of proof for teaching mathematics (Ellis et al. 2012; Lesseig 2016). The five statements varied in difficulty, and included claims from geometry, algebra (problem solving and symbolic manipulation), elementary number theory and infinite geometric series. The first three arguments, pertaining to the sum of the angles in any triangle, were drawn directly from Knuth (2002a, p. 69). The argument regarding how to determine multiples of three was from Knuth (2002b, p. 394). The two arguments dealing with the infinite geometric series and the two arguments related to the transformation of the quadratic equation were adapted from Ellis et al. (2012). Finally, we constructed two arguments justifying a general formula for the *n*th term in a patterning task typical of those often used with students in Years 7–9.

In Part III of the survey instrument, PSMTs were asked which of the 10 arguments from Part II they felt were most helpful for students working on each of the five different statements. PSMTs were asked to make a selection and provide an explanation for their choice(s). This series of questions allowed us to examine what PSMTs attend to when making instructional decisions related to proof. The survey instrument in Australia and the USaA was administered online using Qualtrics, during Semester 1, 2017 (i.e. January–May), and these participants had two weeks to complete the survey instrument. Korean participants completed a written version of the survey instrument in one sitting in March 2017. Although all 34 PSMTs began the survey instrument, students were not required to answer every question in order to progress throughout the instrument. As such, findings are reported with respect to the number of PSMTs who answered each question.

Data analysis

Various frameworks have been developed to delineate mathematical knowledge that would support the work of teaching proof (e.g. Lesseig 2016; Steele and Rogers 2012; Stylianides and Ball 2008). Lesseig's (2016) MKT for Proof framework describes Common and Specialised Content Knowledge related to constructing proof, recognising components of proof and understanding the many functions of proof. As mentioned earlier, the MKT for Proof framework (Table 1) provided an initial lens through which the researchers analysed collected data and grounded any subsequent interpretations of PSMTs' conceptions of proof and their proof evaluations.

Data analysis for this study occurred in two phases. Phase I involved a combined analysis of the data from Australia and the USA, while Phase II included the data from Korea. One reason for this approach was that PSMTs from Australia and the USA responded in English whereas Korean PSMTs responded in Korean. Given the qualitative nature of the data, there was a strong possibility that meaning would be lost in translation.

Part I coding In Phase I, the Australian and US researchers first separately analysed responses of the PSMTs from their own respective countries to the following questions in Part I of the survey instrument: (1) What purpose(s) does proof serve in mathematics? (2) What makes an argument a proof? (3) If proofs are to be taught to students, what are your reasons for teaching? and (4) What will be the constraints, if any, on teaching proofs?

Initially, the researchers analysed the data with a priori codes drawn from literature pertaining to each question. For example, VERIFY, EXPLAIN, SYSTEMATISE, DISCOVER and COMMUNICATE adapted from de Villiers (1990) were used for question 1. For question 2, we created codes to align with the essential proof understandings in Lesseig's (2016) MKT for Proof framework (e.g. LOGIC, THEOREM, GENERAL). After this first pass, the two researchers met virtually to discuss other themes that arose and refined codes to incorporate additional themes (e.g. adding codes for BUILD UNDERSTANDING and ASSESS UNDERSTANDING for questions 1 and 4) and to remove themes that were not applicable to certain questions. On the second pass, both researchers analysed the data from both countries using the agreed-upon codes. Once completed, the researchers met virtually to discuss similarities and differences in their coding The inter-rater reliability (IRR) was 85% for the 60 responses coded in Part I. Based on discussion, consensus codes (reported below) were created for each PSMT response.

In Phase II, the Korean researcher and a research assistant analysed the responses of Korean PSMTs. The researchers extracted keywords from Korean data and set the Korean codes. The researchers then compared the Korean codes with the codes from the Australian and US data analysis. The codes INTEREST and DEDUCTIVE set in the Korean data analysis did not appear in the codes from the Australian and USA. The researchers integrated the code DEDUCTIVE into the code THEOREM and added the code INTEREST for Part I. The IRR within Korean data was 89% for Part I. There were 48 Korean responses coded in Part I.

Parts II and III coding In Phase I coding of Parts II and III, the Australian and US researchers again read all of the responses from PSMTs in their respective countries and then met to discuss themes and categories that emerged. While many of the responses included statements related to mathematical aspects of proof captured in the codes from Part I, PSMTs also attended to features important to teaching such as the extent to which a proof was accessible to students or used multiple representations. Additional codes (i.e. ACCESS, CLEAR, REPRESENT, CORRECT, COMPLETE) were developed to capture these pedagogical considerations. Once agreeing upon these codes, the researchers independently coded all the data from Parts II and III in which PSMTs were evaluating whether or not the 10 student-generated arguments were valid proofs, along with which arguments they thought were most helpful to students when teaching the respective five statements. Independent coding was again followed by consensus coding. The IRR was calculated at 83% for Part II, and 86% for Part III. There was a total of 134 responses coded for Part II, and a total of 50 responses coded for Part III.

In Phase II, the Korean researchers analysed the responses of Korean PSMTs and set the Korean codes and then compared the Korean codes with the codes from the Australian and USA. The researchers incorporated the code INTUITIVE, which appeared only in Korean data analysis, into the code CLEAR set in the Australian and US data analysis. The code INTUITIVE was set in Korean responses such as "The argument is easy to understand because it is intuitive." The researchers concluded that integrating the code INTUITIVE into the code CLEAR was appropriate. The IRR was calculated at 87% for Part II, and 88% for Part III within Korean data. There was a total of 120 Korean responses coded for Part II, and a total of 48 Korean responses coded for Part III.

Results

Part I—PSMTs' conceptions of proof and proof teaching

Questions in Part I of the survey instrument were designed to reveal PSMTs' conceptions of proof and the teaching of proof. In this section, we present findings for each of the open-ended questions and highlight similarities and differences in responses by country. For each of the survey instrument parts, we describe the three most common responses and provide a table indicating overall counts of other salient themes by cohort (see Tables 2, 3, 4, 5).

What purpose(s) does proof serve in mathematics? Participants indicated that proof establishes an axiomatic system to formalise mathematical knowledge, provides verification that a mathematical statement is true (or false), and helps to explain why a statement is true or makes sense. In this way, the notion of proof acting as a "failsafe" during the teaching and learning of mathematics underpinned many statements regarding the purpose of proof.

Nearly half of the participants across all three countries (14 of 33) made reference to how a proof provides verification of the truth of a mathematical statement, aligning with the *Functions of Proof* (CCK) criterion outlined in the MKT for Proof framework (Table 1). For example, Brooke (AUS) responded, "The purpose of a proof in mathematics is in its name; to prove a theorem/argument beyond reasonable doubt". Yuri (KOR) alluded to the dual nature of proof in providing justification in addition to explaining why something was true, stating, "The purpose of a proof in mathematics is to verify a mathematical statement is true and to explain why the statement holds to others".

Table 2 Summary of participant responses: the purpose of proof

Code	Meaning of code	PSMT (AUS)	PSMT (USA)	PSMT (KOR)	PSMT (Total)
VERIFY	To provide verification that a statement is true or false	5	2	7	14
SYSTEM	To establish an axiomatic system, referring to the formalisation of mathematical knowledge	4	7	0	11
EXPLAIN	To explain why a statement is true or makes sense	2	4	1	7
BUILD U	To build or deepen understanding of the mathematical concepts underlying the proof	0	3	4	7
COMM	To communicate mathematics	0	2	0	2
DISC	To discover mathematical theorems	0	0	2	2

Table 3 Summary of participant responses: what makes an argument a proof?

Code	Meaning of code	PSMT (AUS)	PSMT (USA)	PSMT (KOR)	PSMT (Total)
THEOREM	It is based on accepted statements or theorems	2	7	7	16
LOGIC	It follows a logical structure	3	3	3	9
INFALLIBLE	It removes any doubt about the truth or falsehood of the statement	2	3	4	9
GENERAL	It proves the statement in general by covering all cases within the domain	2	2	2	6
VERIFY	It provides verification that a statement is true or false	2	0	0	2

Half of the participants from the USA and Australia (11 of 21) reported that proofs serve to establish an axiomatic system, or that proofs assisted in the formalisation of mathematical knowledge. This finding was expressed in a variety of ways, but ultimately the common element was that proof serves to build logically and deductively on previously established facts. Participants attested to how the foundational nature of mathematical statements assists in moving toward an answer or toward building understanding. For example, Julie (AUS) commented that proof helps "to establish fact, it provides the means where fundamental truths can be established which then provide the foundations for further understanding to be built".

The next most frequently cited purposes of proof were to explain why a statement is true and to build understanding (7 of 33 in each category). Proof as explanation was more prominent in responses from the USA and Australia, with only the above-mentioned Korean response from Yuri coded as EXPLAIN. These assertions align with the *Additional Functions of Proof* (SCK) criterion (Table 1) that proof provides

Table 4 Summary of participant responses: why teach proofs to students?

Created code	Meaning of code	PSMT (AUS)	PSMT (USA)	PSMT (KOR)	PSMT (Total)
T-SKILLS	To teach skills in logical reasoning, argumentation and problem solving	5	5	4	14
BUILD-U	To build or deepen understanding of the mathematical concepts underlying the proof	3	3	5	11
S-AGENCY	To increase or build student agency	2	4	5	11
DISC	To discover or explore idea or create new knowledge	0	3	0	3
ASSESS-U	To assess student understanding	0	2	0	2
VERIFY	To provide verification that a statement is true or false	0	0	2	2
INFALLIBLE	To remove any doubt about the truth or falsehood of the statement	0	0	1	1

Table 5 Summary of participant responses: what are the constraints on teaching proof?

Created code	Meaning of code	PSMT (AUS)	PSMT (USA)	PSMT (KOR)	PSMT (Total)
S-KNOW	Students may not have the requisite mathematical content knowledge or skills for proof	3	4	7	14
TIME	Takes too much time to teach properly	2	4	4	10
CURRIC	There is not enough clear direction in standards or curriculum	3	3	0	6
ABSTRACT	Proof is too abstract for adolescents	0	3	2	5
T-KNOW	Teachers may not have the knowledge and skills necessary to teach proof	1	2	0	3
INTEREST	Students are not interested in learning proofs and do not want to learn proofs.	0	0	3	3

insight into why the statement must be true. For example, participant testimony included statements that proofs explain and "provide evidence for mathematical 'rules'" (Helen, AUS) or "give reason for why something is what it is" (Paige, USA). Two participants went further to explicitly state that proofs that explain why were valuable precisely because they help to build understanding—another *Additional Functions of Proof* (SCK) criterion. Meanwhile, four Korean participants stated that the purpose of proof was to understand mathematical theorems. One of them connected understanding with the ability to memorise a theorem, "by examining how a theorem came out through the proof, we can better understand the theorem and memorise it for a long time" (Sooji, KOR).

What makes an argument a proof? For the second survey instrument question, participant responses aligned closely with the criteria located in the *Essential Proof Understandings* (CCK) domain of the MKT for Proof framework. Those criteria stipulate that a theorem has no exceptions, a proof must be general, a proof is based on previously established truths, and the validity of a proof depends on its logic structure. The most frequently elicited responses were that a proof is based on accepted statements, follows a logical structure, and thusly removes any doubt about the veracity of the statement.

Half of the participants (16 of 32) asserted that an argument must be based on accepted statements or theorems. Responses coded as THEOREM included explicit reference to theorems or previously established facts as in Vicky's (USA) response that a proof is "based on prior statements that have been proven or accepted as true". Participants who noted the importance of basing proofs on known facts often included this in reference to the logical structure of arguments. According to Oliver (USA), a proof "contains a logical progression as well as axioms. A proof must also seek to make a claim that can be directly supported by a logical argument that is fundamentally based on axioms". The third most frequently cited criterion was that a proof must remove any doubt about the truth or falsehood of a mathematical statement. For instance, Yuri (KOR) stated that "proof should be able to show everyone without any doubt that a mathematical statement is true".

If proofs are to be taught to students, what are your reasons for teaching? According to the proffered responses for the third survey instrument question, participants feel that proof should be taught to students to impart a variety of mathematical skills such as logical reasoning, to build an understanding of the mathematical concepts underlying the proof and to increase student agency.

Nearly one half of the participants (14 out of 31) expressed how proofs serve as an instructional tool to build student skills in relation to broader mathematical processes including logical reasoning, argumentation and problem solving. These reasoning skills were often seen as essential in areas outside of mathematics as illustrated in Naomi's (USA) response:

> … teaching proofs to students would help students to develop logical reasoning, which is applicable beyond the study of mathematics. It will prepare students to recognise when something is presented to be true but isn't proved to be true, or when something is argued to be true, but there happens to be a flaw in the logic of the argument.

Participants from all three countries mentioned the role of proof in promoting problem solving. For example, Venny (KOR) stated, "Students can develop their ability to apply a theorem to problem solving if they have a good understanding of proof of a theorem". In similar fashion, Alex (AUS) highlighted how proof "promotes problem solving skills and is essential to understanding where formulas come from".

About a third of participants (11 of 31) described how proofs help to build or deepen student understanding of the mathematical concepts underlying the proof. Helen (AUS) linked her own experiential learning of proofs to the educative benefits afforded to students:

> When I did proofs in school it enhanced my understanding of mathematical rules and concepts. Conducting proofs is quite abstract/requires high level thinking and hence, increased my conceptual understanding of topics, rather than surface-level rote learning.

Three Korean participants made further reference to how students do not have to memorise a theorem if they understand proof of it. For example, Seoyun (KOR) commented, "I think teaching proofs can help students understand mathematical theorems rather than memorising them".

Increase student agency Approximately one third of participants (11 of 31) expressed how teaching proofs increases or builds students' sense of self-agency. PSMTs' statements about student agency were often made in conjunction with references to other skills proof afforded to students (e.g. problem solving and argumentation). For example, Rose (USA) outlined how proof instruction can assist in fostering a sense of deeper, more independent learning:

> I believe that students who feel confident in their understanding and production of proofs will be able to use mathematics to their fullest ability. Being able to write a proof requires spelling out properties and known information in a way that

> allows a student to reach new conclusions. Students that understand proof can break free of memorisation and exercise agency in problem solving, which supports the ultimate goal of independent learning.

Similarly, Minju (KOR) claimed, "Learning proofs would prevent students from learning mathematics through rote problem solving, or formula memorisation".

What will be the constraints, if any, on teaching proofs? Most respondents asserted that proof is a difficult topic, skill or body of knowledge to teach. Participants also spoke of the heavy time investment and the curriculum as constraints for teaching proof.

Participants invariably mentioned how proofs are difficult for students to learn. Learner-centred constraints focussed on a perceived inability for adolescents to reason due to insufficient mathematical knowledge or the abstract nature of proof in general. This response from Yongjun (KOR) is typical of those in which PSMTs expressed concern over students' mathematical content knowledge: "When we prove any proposition, we should use the theorems that have already been proven. However, it is often the case that students do not know well the already proven theorems". Participants also questioned the developmental appropriateness of teaching proof to secondary students. In these responses, PSMTs spoke more generally about how proof itself was too abstract for adolescents or commented that proof teaching was constrained by "students' abilities and the language surrounding proof" (Julie, AUS).

Participants also spoke of the considerable time required to teach proofs properly to students. PSMTs voiced concerns over whether there was enough time "for students to learn proofs faithfully" (Hyojin, KOR). Entwined within these statements were some of the previously mentioned complexities of teaching proof, such as limitations in student prior knowledge and the inherent abstractness of proof. Helen (AUS) commented, "Many students do not grasp them conceptually very well so [teaching proofs] may be time-consuming, or may lower students' self-esteem on a topic".

Part II—PSMTs' evaluations of student-generated proofs

The purpose of Part II in the survey instrument was to determine the extent to which PSMTs felt that 10 student-generated arguments constituted a proof, and to ascertain what features of proof PSMTs attended to when making that determination. Across all of the arguments, the most common comments were that these arguments proved the statement in general were based on accepted statements or theorems, and followed a logical structure. To a lesser degree, PSMTs argued (in the affirmative or the negative) that the proofs were valid, used multiple representations, were easy to follow or understand and contained all of the steps.

We discuss these findings according to three themes that resonate with previous literature and are particularly relevant to knowledge for teaching proof: PSMTs' understanding of generality; the interpretation and acceptance of non-symbolic arguments; and additional characteristics that support student learning. Results of the acceptance rate, whether each of the 10 student-generated arguments is a proof or not, are displayed in Table 6.

Table 6 Participants' evaluations of student-generated arguments

	AUS Yes	US Yes	KOR Yes	Total Yes	AUS No	US No	KOR No	Total No	Percent Yes
Andrew	2	4	2	8	6	6	10	22	27
Mark	8	8	12	28	0	2	0	2	93
Jane	1	7	8	16	7	3	4	14	53
Minna	5	4	3	12	2	6	9	17	41
Kelly	3	5	3	11	4	3	9	16	41
Jisoo	3	7	8	18	4	1	4	9	67
Anny	3	3	6	12	2	2	6	10	55
Zack	0	0	0	0	5	5	11	21	0
Amy	2	1	5	8	3	3	7	13	38
Beth	4	3	11	18	1	1	1	3	86

Generality PSMTs attended to whether (or not) arguments considered the general case in nearly 30% of their evaluations. The ways in which PSMTs' considered generality are best illustrated in their responses to the set of three arguments concerning the sum of the angles of a triangle from Andrew, Mark and Jane and to Minna's divisibility argument. About two thirds (22 of 30) of all participants indicated that Andrew's argument, which involved tearing the corners off triangles and putting them together to form a straight line, did not constitute a proof. The most common reason given was that the argument rested on only three specific cases, and therefore lacked generality for all triangles. In contrast, participants indicating that Andrew's argument *did* constitute a proof tended to make this assertion on the basis that it covered all cases—and subsequently, these were also characterised by generality. For example, Grant (AUS) concluded that Andrew's argument "shows that for all cases of triangles (obtuse, right, acute) the sum of the angles equals 180 degrees. Perhaps this needed to be stated in the proof, that these three types of triangles cover all the possible cases".

An overwhelming number of participants (28 of 30) agreed that Mark's argument was a proof. The most common rationale was that this argument demonstrated generality, as illustrated in Rose's (USA) comments:

> This proof relies on the fact that 'any triangle' has a base, and that we can always draw a line that is parallel to the base that goes through the third point of the triangle. This is true of all triangles, and the rest of the proof follows from proven theorems or facts about parallel lines and angles. Thus I believe this is a proof of the statement because its procedure is generalisable to 'any triangle'.

Participants appeared divided as to whether Jane's argument constituted a proof (16 = Yes, 14 = No). Participants who did consider Jane's argument a proof highlighted how the reasoning could apply to many other cases. While fewer PSMTs judged Jane's argument as not a proof, the reasons for non-acceptance were most often associated with clarity, as some PSMTs were unable to follow the argument.

Participants also appeared divided as to whether Minna's *Divisibility* argument constituted a proof (12 = Yes, 17 = No). Those who agreed that Minna's working through a specific example (i.e. using 756 to demonstrate that if the sum of the digits of a number is divisible by 3, then the number itself is divisible by 3) was a proof commented chiefly on the generality of the argument. This response from Changwoo (KOR) was typical, "Although it is a proof with a specific number, it is generally verifiable in the same way for all numbers". Participants avowing that Minna's argument did not constitute a proof stated that it lacked generality and that errors were made in mathematical reasoning. For instance, Helen (AUS) highlighted that "It was explained with one specific example and did not use universal terms that related to all situations (e.g. n and m)". Difficulties PSMTs had in determining whether Minna had provided a valid, general, proof lead into our next theme related to the ability to interpret or accept non-symbolic arguments.

Interpretation and acceptance of non-symbolic arguments Participants were quite divided in their acceptance of two non-symbolic arguments provided in the survey instrument, Minna's generic example proof discussed above (Mason and Pimm 1984) and Kelly's *Infinite Series* proof. Only 41% of PSMTs accepted these two arguments as proofs, and Korean participants were even less likely to adjudge in the affirmative (only 25% in each case).

Approximately 60% of PSMTs concluded that Kelly's pictorial argument for the sum of an infinite series did not constitute a proof. The chief reasons in this case included a lack of logical reasoning, explanation or theorems rather than a lack of generality. For example, Seoyun (KOR) commented, "There is a lack of explanation as to how the picture is drawn, and the statement that the picture proves the conclusion seems to be lacking. Other participants expressed how the stand-alone use of a visual representation did not provide a compelling or convincing argument:

> This visual proof does not convince me because I imagine zooming in on the upper right corner forever. Although it will appear to fill up an area (approach a value) of 1, I can forever zoom in and the square remains incomplete (Rose, USA).

PSMTs who agreed that Kelly's argument constituted a proof tended to do so with some scepticism. For instance, Helen (AUS) suggested that although the argument ostensibly is clear and logical, it suffered from a lack of explanation, "This makes sense and is clear. However, I am not sure about the use of no words. Also doesn't use any rules to explain it".

Part III—PSMTs' selection of proofs to use in teaching

Our intent in Part III of the survey instrument was to identify features of proof that PSMTs attend to when deciding which arguments to present in a classroom. Our analysis revealed that PSMTs not only valued arguments that were valid proofs (i.e. correct), but also considered other pedagogical features such as whether the proof was clear, accessible to students, and might support student learning. The mode of representation and clarity of the argument were the two most commonly mentioned characteristics upon which PSMTs based their decisions, evidenced in approximately 15 and 25%, respectively, of the responses.

Mode of representation PSMTs from all three countries noted how specific representations such as visual arguments and formulas would benefit student learning. For example, when choosing arguments for the sum of angles, Cole (AUS) noted how Andrew's visual method, "would work best to initially demonstrate the reasoning to all levels of student ability", and that Mark's answer "would be used to extend students' thinking". Helen and Daeun similarly saw value in presenting more than one of the infinite series arguments:

> I think both methods [Kelly and Jisoo's] could be useful because it is good to see things visually and also through working. There are multiple ways to do proofs and students may benefit from being taught both ways (Helen, AUS).

> Kelly's argument is a visualised proof that it has an advantage of understanding easily at a glance, and Jisoo's argument is clearly understood as a proof through using formula (Daeun, KOR).

Statements about it being important for students to see different ways of thinking and proving, as demonstrated in Helen's response, were fairly common.

Clarity and accessibility of the argument PSMTs also valued arguments that were simple and easy to follow. This was mostly evidenced in general comments about how a particular argument was "easy to understand intuitively" or "not too complicated". However, PSMTs sometimes expanded on the importance of clarity or simplicity to consider whether students themselves would be able to make sense of the argument. The need for proofs to be accessible was expressed in Cole's rationale above that Andrew's method would demonstrate reasoning "to all levels of student ability" and in Lana's response below.

> Andrew's answer is accessible and provides the opportunity to explore many possible cases. Have each student make their own triangle and recreate Andrew's proof by tearing them up. With some scaffolding, Mark's proof can be accessible to students as well, but care must be taken in identifying the alternate interior angles (Lana, USA).

Other responses related to access included those in which PSMTs speculated on the background knowledge and experiences of their (future) students. Hyonsu (KOR) chose Mark's argument for the triangle sum because "I think Mark's argument is most helpful because it would be familiar for students to use the properties of the alternate angles in parallel lines". Alternatively when talking about the arguments for the infinite series, Lana (USA) responded that while both Jisoo and Kelly's arguments would be helpful to a group of students working on infinite geometric sequences, she would "exercise caution in Jisoo's argument", reasoning that "It is a valid proof, but student understandings of geometric series can make his proof confusing". Statements like this, while not widespread, indicated how some PSMTs attended to developmental aspects of proof—a key component of pedagogical knowledge for teaching proof (Lesseig 2016).

Additional characteristics Although 10% of PSMTs responses specifically attended to the correctness of the argument, this did not mean that PSMTs would only present arguments they felt constituted valid proofs. In fact, Walter (USA) reasoned that it would be valuable to show Amy's argument for the toothpick problem even though he did not consider it a proof "because it showed the steps Amy took but not why it works". Walter's rationale was that "It's still good to see some non-examples. Also, Amy's argument contains some good mathematical ideas that the class could draw upon together". Changwoo (KOR) also stated that "Jisoo's argument is a valid proof, and Kelly's argument (though not a proof) has the advantage that students can approach it intuitively". Finally, despite the fact that generality and logic were common features, PSMTs from all three countries attended to when evaluating proofs in Part II, only the Australian and Korean PSMTs made these characteristics explicit in their responses in Part III. Julie's (AUS) choice to share Mark's argument demonstrates how these two characteristics were often intertwined: "Andrew's argument was too simple in only using one triangle and Jane's argument was too complicated. Mark's deduces the proof in an easy to follow and logical manner".

Korean PSMTs who paid attention to the logic and generality in their responses in Part III showed a tendency to consider students' understanding together. They stated that it is necessary for proofs learning that students experience both intuitively understandable arguments and logical arguments. For example, in considering which infinite series arguments to present to students, Yongjun (KOR) commented, "After students intuitively understand the results through Kelly's argument, it would be good to prove the logical validity of the results through Jisoo's argument".

Discussion

This research had two specific aims. The first aim was to investigate how PSMTs in the USA, Australia and South Korea perceived the purpose of proof in the context of secondary mathematics teaching and learning. The second aim was to determine what characteristics of proof PSMTs centralise in their evaluations of proofs. A discussion of the findings for each aim is offered below.

PSMTs' perceptions of the purpose of proof in secondary mathematics teaching and learning

PSMTs from all three countries emphasised the verification role of proof, describing proof as the means for establishing mathematical truth and demonstrating that a result is true beyond a reasonable doubt. According to previous research (e.g. Knuth 2002a; Varghese 2009), this dominant view of proof has the potential to limit teachers' perspectives on when and how proof should be used in a secondary mathematics context. In our study, however, such a view was not considered a deficit but rather as a common starting point. To illustrate, a number of PSMTs also acknowledged the explanatory role of proof (Hanna 2000) and recognised proving as a venue for students both to build and communicate understanding. Explicitly discussing verification and explanation as complementary roles of proof with preservice teachers (Bleiler-Baxter and Pair 2017) might broaden PSMTs' views of proof and their ability to integrate proof more consistently into their future teaching.

When responding directly to questions about critical features that make an argument a proof, fewer than 20% of PSMTs made reference to the fact that a proof must be general—one of four essential understandings comprising CCK for proof teaching. However, the majority of PSMTs were able to recognise generality as a necessary component when evaluating proofs in Part II of the survey instrument. Furthermore, in their evaluations, PSMTs demonstrated a robust understanding of proof from a mathematical perspective, and made explicit statements that examples did not constitute proof. This understanding is particularly notable in light of prior research documenting students and preservice teachers' acceptance of empirical arguments as proof (Harel and Sowder 2007; Simon and Blume 1996; Stylianides and Stylianides 2009).

While survey instrument responses to Part I generally conveyed a degree of accord across the three countries, there were some key differences worth highlighting. For instance, over half of the Australian and US participants stated that proofs serve to establish an axiomatic system or assist in the formalisation of mathematical knowledge. However, none of the Korean participants mentioned systemisation. There are two possible reasons why Korean PSMTs did not mention systematisation as a purpose of proof. First, Korean PSMTs are trained in a Department of Mathematics Education within a College of Education and as a result take on a teacher perspective from the beginning. Although Korean PSMTs have learned a great deal of upper-level mathematics and are exposed to proofs in the College of Education, they seem to look at proof in an educational context rather than a mathematical context. Second, Korean PSMTs learned proofs for the first time in Year 8 when they were junior high school students. The mathematics textbook they used provided this definition: "A proof is to verify the truth of a certain proposition by using definitions or true statements which are already known, not by using experiment or measurement" (Jung et al. 2011, p. 198). In this definition, the purpose of proof is to justify the truth of a proposition. It therefore makes sense that Korean PSMTs predominantly mentioned verification as the purpose of proof.

A lack of pre-requisite student knowledge and insufficient allocations of time were prominently identified as constraints for teaching proofs by PSMTs in all countries. In addition to claims that insufficient student knowledge impedes effective teaching of proofs, participants commonly asserted that proofs are complex to learn and questioned the developmental appropriateness of teaching proofs to adolescent learners. This finding aligns with the work of Bergqvist (2005), whose results indicated that teachers tended to underestimate students' reasoning levels, and that teachers believed that only a small group of students could use higher-level reasoning in mathematics. In a similar vein, Varghese (2009) found that while PSMTs acknowledged that learning proof could improve students' problem-solving ability and logical reasoning skills, they generally deemed teaching proofs as a waste of time.

Across all countries, assertions of proofs requiring considerable time to teach well were often tied to statements of student prior knowledge. Despite these cultural similarities, there were two distinct differences espoused by Korean participants compared to their Australian and US counterparts. First, only the Korean participants mentioned student interest as a teaching constraint. From a Korean perspective, this is a significant point as it highlights how the PSMTs may be attending to emotional aspects of proof learning. To amplify this finding from a cultural perspective, Korean students have shown a very high level of mathematical achievement in international student assessments such as TIMSS and PISA. However, Korean students showed a very low index in emotional aspects, including interest in mathematics learning, and displayed negative attitudes toward mathematics. Students' low interest in mathematics has been recognised as an important issue to be addressed nationwide in Korea. It seems that Korean PSMTs are well aware of Korean students' low interest in mathematics as a social issue. Second, curriculum is not mentioned as a teaching constraint by Korean PSMTs. Korean PSMTs learned many proofs when they were in junior high school. Contrary to the PSMTs in this study, current Korean students do not learn proofs in junior high school due to the weakening of proof in the Korean mathematics curriculum since 2009. As such, the mathematics curriculum in Korea has become a stumbling block to teaching proof. However, Korean PSMTs are not aware of these changes, and therefore, unlike PSMTs in the USA or Australia, they do not refer to the curriculum as a constraint on teaching proofs. PSMTs from Australia and the USA are not necessarily aware of the curriculum they will be expected to teach. However, PSMTs in both countries may be influenced by years of exposure to an assessment-driven culture imbued with messages that content (e.g. memorising formulas and following procedures) matters more than exercising proof and reasoning techniques (Ginsburg et al. 2009; Weiss et al. 2003).

Central characteristics of PSMTs' proof evaluations

The characteristics PSMTs attended to when evaluating arguments did not necessarily map directly onto those characteristics PSMTs used when deciding which proofs they might present to students. For instance, when asked to identify necessary components of a proof in Part I of the survey instrument, PSMTs focused on the requirement that proofs be based on established facts and follow a logical progression, and subsequently included statements about the logical structure or use of theorems in their rationales for the acceptance (or not) of the student-generated arguments as proof in Part II, thus demonstrating a clear alignment. In contrast, nearly 30% of PSMTs' proof evaluations attended to whether arguments considered the general case, whereas in Part I generality was mentioned by less than 20% of participants. Responses to Part III also indicated the value PSMTs place on using multiple representations to make proofs more accessible to a range of students, characteristics not mentioned in the initial proof evaluations. Differences across survey instrument sections highlight the importance of considering PSMTs' actions across multiple settings. From a research perspective, these findings suggest that measures of PSMTs' proof conceptions should be situated in teaching tasks (Steele and Rogers 2012).

Additionally, the researchers noted that merely knowing that a proof must be general was not necessarily sufficient, as PSMTs had different interpretations of what constituted generality. This finding was most evident in PSMTs' assessment of two "non-traditional" arguments included in the survey instrument, one a visual argument, and the other a generic example proof (Karunakaran et al. 2014; Mason and Pimm 1984). This finding aligns with the work of Inglis and Alcock (2012) who not only concluded that mathematicians' standards of proof validity differ, but also questioned whether students are receiving a consistent message about what constitutes a valid proof. Given their accessibility to students, visual arguments and generic example proofs have the potential to "bridge the gap" from empirical toward more deductive modes of argumentation (Karunakaran et al. 2014). Thus, the researchers contend that increasing PSMTs' proficiency with constructing and assessing generic example proofs and visual arguments is an important step toward enhancing the role of proof in secondary mathematics classrooms.

Limitations

There were at least two limitations for the findings of this pilot study. First, there was a relatively small number of participants ($n = 34$) involved across the three different contexts which delimited the generalisability of the results. Nevertheless, and especially given the paucity of related research in the literature base, the findings of this pilot study are valuable insomuch as preliminary claims about PSMTs and proof education across those contexts can be made. Furthermore, the study facilitated a demonstration of the utility of the survey instrument within a context of preservice teacher training. Second, the authors acknowledge that the design of the study did not allow any claims to be made about PSMTs' instructional practice in regard to proof (i.e. there were no opportunities to observe participants teaching proofs in classrooms, or interacting with students). Despite this acknowledgement, we claim that the type(s) of proof knowledge PSMTs possess and how teachers hold that knowledge determines to a large extent the nature of proof instruction and the opportunities they will be able to provide students. For instance, we contend that if PSMTs are unsure about the status of visual proofs, they are less likely to promote multiple argument representations to a class of secondary students. Conversely, those PSMTs who are able to recognise the features of proof that make it general will be better positioned to help students transition from empirical reasoning to deductive proof.

Conclusion

This pilot study investigated how PSMTs in the USA, Australia and South Korea conceived of proof and proof teaching. Our study revealed striking similarities in proof understandings and characteristics of proof PSMTs in the three countries attended to in their evaluations of student-generated proofs. Our hope is that this study might inform further research and practice regarding the preparation of secondary mathematics teachers. First, we encourage teacher educators to design content and activities that build on PSMTs' strengths and enable PSMTs to productively coordinate disciplinary components of proof (most often learned in mathematics content courses) with pedagogical knowledge of proof, such as developmentally appropriate representations. Secondly, our findings related to possible cultural differences (e.g. Korean PSMTs' concern over maintaining student interest in mathematics) might lay the foundation for future cross-country comparative studies.

Appendix 1

Table 7 Questions for survey instrument Parts I, II and III

	Main questions	Sub-questions
Part I	What purpose(s) does proof serve in mathematics? What makes an argument proof? Do you think that proof should be taught to students? If proof is to be taught to students, what are your reasons for teaching proof? To what degree will you plan lessons for fostering students' proof ability? What will be the constraints, if any, on teaching proof?	Likert of agreement: strongly agree, agree, disagree, strongly disagree Likert of frequency: all the time, frequently, when opp. arise, rarely
Part IIa	Evaluate 3 student arguments attempting to prove: The sum of the angles in any triangle is 180 degrees.	Is Andrew/Mark/Jane's argument a proof? Y or N. Please elaborate on why you said yes or no (for each argument).
Part IIIa	Which of these arguments (if any) is most helpful for a group of students working on the sum of the angles in any triangle?	Explain your answer.
Part IIb	Evaluate 1 student argument attempting to prove: If the sum of digits of a whole number is divisible by 3, then the number itself is divisible by 3.	Is Minna's argument a proof? Y or N. Please elaborate on why you said yes or no.
Part IIc	Evaluate 2 student arguments attempting to prove: $12 + 14 + 18 + \cdots = 1$	Is Kelly/Jisoo's argument a proof? Y or N.
Part IIIc	Which of these arguments (if any) is most helpful for a group of students working on infinite geometric sequences and series?	Please elaborate on why you said yes or no (for each argument). Explain your answer.
Part IId	Evaluate 2 student arguments attempting to prove: $x^2 + ax = (x + a^2)^2 - (a^2)^2$	Is Anny/Zack's argument a proof? Y or N. Please elaborate on why you said yes or no (for each argument).
Part IIId	Which of these arguments (if any) is most helpful for a group of students working on the proof problem?	Explain your answer.
Part IIe	Evaluate 2 student arguments attempting to prove: $3n + 1$ can be used to represent the total number of toothpicks needed to construct the pattern for n squares (connected, in a row).	Is Amy/Beth's argument a proof? Y or N. Please elaborate on why you said yes or no (for each argument).
Part IIIe	Which of these arguments (if any) is most helpful for a group of students working on the proof problem?	Explain your answer.

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