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Generalisation through noticing structure in algebraic reasoning

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capital MATHS

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Edited by Valerie Barker,
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CONTENTS

challenge us, to support change and growth, and to affirm the profoundly important work of all in education.

Part of that challenge to me, and I hope to all at the conference, is how we are able to take these ideas and challenges back to our own communities—to lead learning, innovation, influence, change and growth. My hope is that our conference experiences guide us in the leadership that we take to our own communities, allowing us to reflect on and reinvigorate, validate and reappraise, challenge and change our personal and collegial practices. We are a truly supportive community of each other as educators, whether it be as delegates at this conference or in our own workplaces and professional communities. It is appropriate for me to acknowledge with much gratitude the support and guidance I have been given in this editorial role, from many whom I have not met personally, as a particular example. It heartens and encourages me, as I am sure it will for all delegates, that in my everyday practice I am able to influence, lead and be guided by passionate, caring and committed educators; it is we small investors who can play a profound role in mathematics education as a critical component of securing a capital investment in our nation's future.

As this proceedings' lead editor, I would particularly like to thank Kate Manuel and Toby Spencer (AAMT) for their support and guidance throughout the editorial process.

Valerie Barker, Proceedings Lead Editor

Review process

Presentations at AAMT 2017 were selected in a variety of ways. Keynote and major presenters were invited to be part of the conference and to have papers published in these proceedings. A call was made for other presentations in the form of either a seminar or workshop. Seminars and workshops were selected as suitable for the conference based on each presenter's submission of a formal abstract and further explanation of the proposed presentation.

Authors of seminar and workshop proposals were also invited to submit written papers to be included in these proceedings. These written papers were reviewed without any author identification (blind) by at least two reviewers. Reviewers were chosen by the editors to reflect a range of professional settings. Papers that passed the review process have been collected in the 'Professional Papers' section of these proceedings.

The panel of people to whom papers were sent for review was extensive and the editors wish to thank them all:

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GENERALISATION THROUGH NOTICING STRUCTURE IN ALGEBRAIC REASONING

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"Mathematicians see generalising as lying at the very heart of mathematics" (Mason, Graham & Johnston-Wilder, 2005, p. 283). The Australian Curriculum: Mathematics develops number and algebra together as they complement each other. Developing number and algebra together provides opportunities for searching for patterns, conjecturing and generalising mathematical relationships. Further, it allows the focus to be on the process of mathematics and noticing the structure of arithmetic and our number system, rather than the product of arriving at a correct answer.

Algebraic reasoning

Algebraic reasoning underpins all mathematical thinking, as it allows us to explore the structure of mathematics. It pervades all of mathematics and is about describing patterns of relationships, generalising mathematical ideas and identifying mathematical structures (Ontario Ministry of Education, 2013; Van der Walle, Karp & Bay-Williams, 2010). Kaput and Blanton (2005) defined "Algebraic reasoning [as] a process in which students generalse mathematical ideas from a particular set of instances, establish those through the discourse of argumentation and express them in increasingly formal and age appropriate ways." (p. 99)

Focusing on algebraic reasoning alters the study of number and operations from a focus on finding numerical answers to arithmetic problems, or a product approach, to providing opportunities for discovering patterns, conjecturing and generalising mathematical relationships, a process approach (Schoenfeld, 1987; Siemon, Beswick, Brady, Clark, Faragher & Warren, 2015). It is the patterns that provide insights into the structure of mathematics. Noticing the structure of arithmetic forms the foundation of algebraic understanding. Continual development on recognising pattern and structure has been seen to have a positive influence on overall mathematical achievement and builds a stronger foundation for algebraic reasoning (Mulligan, Mitchelmore & Prescott, 2006). A deep understanding of numbers, operations and the relationships between them is necessary for the development of number and algebra sense and an acute sense of number, along with an appreciation of pattern and relationships are necessary requirements for deep mathematical understanding (Siemon et al., 2015).

The big ideas of algebraic reasoning

It can be seen that patterns are at the core of algebraic reasoning. Searching for patterns is a process natural to people (Mason et al., 2005; Siemon et al., 2015). The study of patterns in schools generally begins with repeating patterns, moving onto growing patterns and investigating and employing patterns in the number system. The study of patterns is necessary prior to the development of functional thinking, which focuses on the relationships between two or more varying quantities. Another critical idea in algebraic reasoning is the notion of equivalence. Equivalence is usually represented by an equal sign, an important, but poorly understood, symbol in mathematics. In order to comprehend the notion of equivalence, students need to understand that the equal sign represents a balance on either side, rather than meaning find the answer. Generalisation encompasses all of these big ideas, as we can generalise about pattern, about equivalence and about function and generalisation lies at the heart of algebraic reasoning (see Figure 1).

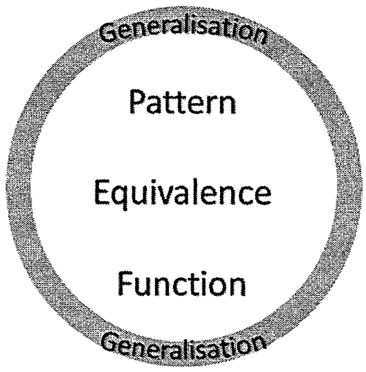

Figure 1. Big ideas of algebra.

Reframing Mathematical Futures II (RMFII) is a three-year project funded by the Australian Government Department of Education and Training under the auspices of the Australian Mathematics and Science Partnership Programme (AMSPP). The project is working with industry partners and practitioners in each State and Territory and the Australian Association of Mathematics Teachers (AAMT) to build a sustainable, evidence-based, integrated learning and teaching resource to support the development of mathematical reasoning in Years 7 to 10. The data collected in the algebraic reasoning component of the RMFII Project has reinforced pattern and function, equivalence and generalisation as the big ideas of algebraic reasoning (Day, Stephens & Horne, in press).

Generalisation

The process of generalisation is about noticing structure. Mason et al. (2005) stated that even very young children can generalise and specialise when they first come to school. While generalising is natural, students need time to notice that they have this sense of generality and they need opportunities to practise, strengthen and extend this natural ability to generalise. Asking students questions about what they notice, whether they can see any patterns and how they are making sense of the mathematics is important.

Often, in order to try to make sense of the mathematics, students will specialise. Specialising may take the form of trying several numbers to see what is happening in a problem. This is a natural approach to mathematical thinking (Mason et al., 2005; Siemon et al. 2015) and it helps students in sense-making while collecting data about a problem. Sense-making is easier if the problems are set in meaningful contexts, as the context allows students to relate what they are seeing back to a specific context. Familiar contexts can be presented using concrete materials, with diagrams and with numbers. Using a concrete-representational-abstract (CRA) approach with students has been shown to be effective (Mudaly & Naidoo, 2015; Sousa, 2008; Witzel, Mercer & Miller, 2003)

Once students are alerted to the idea that patterns are important and they begin to notice patterns, they are in the thinking process of generalisation (Siemon et al., 2015). Questioning students about what they notice, what changes and what stays the same, is important for students to start recognising the structure of the patterns. Eventually students will become attuned to the fact that what changes are variables and the things that stay the same are constants. Other questions that should be routinely asked of students are whether what they have identified always works and in all cases and for all operations (Cooper & Warren, 2008; Kaput, 1999). Much of algebraic reasoning is about searching for, describing, generalising and justifying patterns (Steen, 1988).

Providing students with the time to think, form and try to articulate generalisations to themselves before sharing with a small group or the whole class is essential (Mason et al., 2005) if students are to be confident in articulating ideas. It should be noted that students often do not attend to the same things as their teachers. They see things in different ways and it is the role of the teacher to listen carefully to student explanations about how they 'see' a problem and acknowledge and celebrate articulations and demonstrations that are correct, but not necessarily the same way the teacher 'sees' a problem. Experience with generalising in different contexts may lead to multiple expressions of the same thing (Mason et al., 2005). This, in turn, can often lead to an investigation of equivalent expressions.

Tasks that assist students to notice structure and generalise

Many problems that are designed as arithmetic problems for young students can be extended into generalisations. The following question (Figure 2) was designed for a Year 2 class as a problem-solving question.

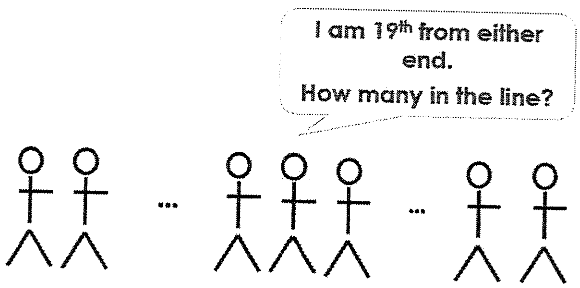

Figure 2. Line up question (adapted from Lovitt & Williams, 2015).

Rather than just seeking the answer to this question, although seeking solutions is useful, there is an opportunity to take this question further. After students have had the opportunity to solve this problem, they can be asked to explain how they 'saw' the problem and draw pictures to represent what their visualisations were. An example of the three ways most students visualise this problem are included in Figure 3. This is an important step for students to see important ideas emerge: that different students visualise problems in different ways, that there are several ways to arrive at a correct answer and that there are multiple ways to write equivalent expressions for the same problem.

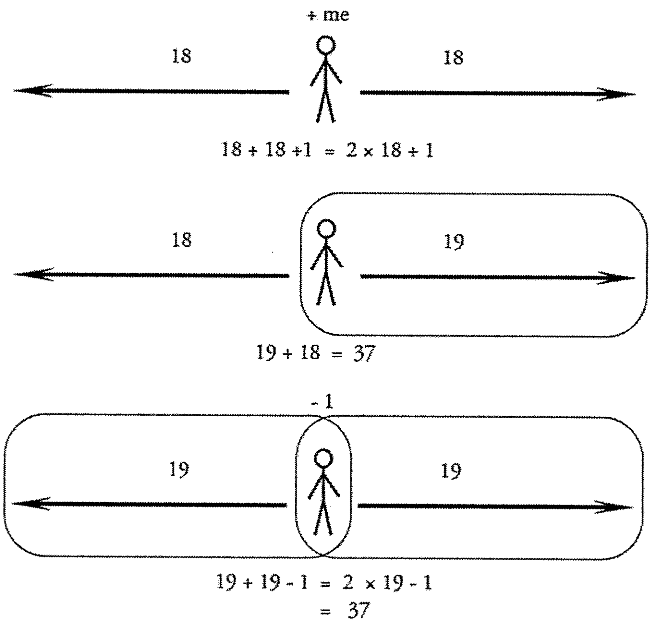

Figure 3. Line up visualisation pictures.

Students may then be asked to use the visualisation that makes sense to them to work out how many students would be in the line if they were 100th from either end and explain in words and/or pictures the process they went through to work out the answer. If students are able to do this successfully they may be asked to generalise the situation in words, pictures and/or symbols (depending on their readiness for symbolic work).

The three generalisations from the visualisations, in order, are:

$$t = 2(n - 1) + 1$$
$$t = n - 1 + n$$
$$t = 2n - 1$$

Interestingly, the third visualisation, which provides the generalisation in the simplest form is the one that the fewest students nominate as their preferred visualisation. Overwhelmingly students 'see' this problem as the first visualisation. That suggests that we should allow students, at least initially, to generalise problems as they visualise them and not always insist that algebraic expressions are in their simplest form, as the expressions need to make sense to the students. Eventually students will be expected to be able to move flexibly between equivalent expressions and express them in their simplest form, but we may rush students to this stage without recognising that they need to first make sense of the problem within its context.

Several arithmetic worded problems can be modified to encourage students to notice the structure of the mathematics. For example, a typical textbook question may read

> Abbey is 140 cm tall. Ben is 4 cm taller than Abbey and Abbey is 6 cm shorter than Charlie. How tall are Ben and Charlie?

By taking out the initial piece of information that Abbey is 140 cm tall, the question may be changed to encourage students to notice the structure and relationships contained in the question:

> Ben is 4 cm taller than Abbey. Abbey is 6 cm shorter than Charlie. Draw a picture showing Abbey, Ben and Charlie's heights. Explain what the 4 and the 6 represent. Try to express these height comparisons in other ways. (Adapted from Carraher, Brizuela & Schliemann, 2000)

By removing the information of how tall Abbey is, the students are forced to consider the height comparisons rather than just perform two computations. This can be taken even further with the introduction of the n-number line (Figure 4), which pays attention to the structure of number lines.

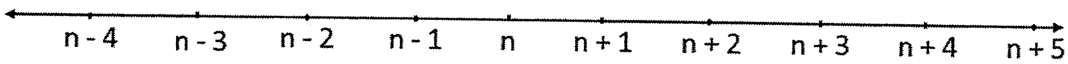

Figure 4. n-number line.

Now the question can be asked if Abbey is n cm tall, position Ben and Charlie's heights on the n-number line. What about if Ben were n cm tall, where would Abbey and Charlie be positioned? If Charlie were n cm tall, where would Abbey and Ben be positioned?

Visual growing patterns are another good way of helping students to notice structure. A great deal of mathematics can be mined from even simple growing patterns. One idea is to use triangles that are used in the construction of bridges and other structures. The simplest of these is known as a Warren truss and is pictured in Figure 5.

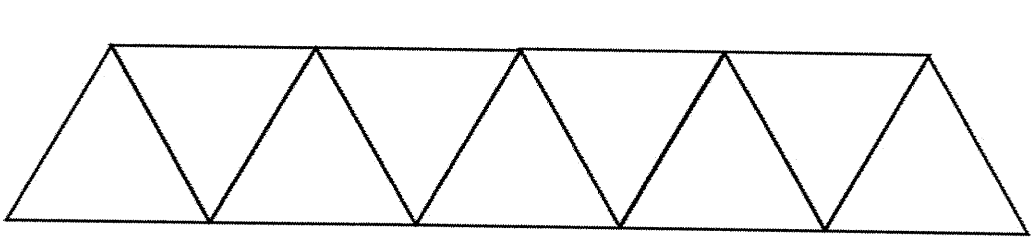

Figure 5. Warren truss.

Students should be encouraged to build this model using popsticks so that they can manipulate the model and show the structure they are considering by physically moving pieces of the model. Questions such as "What changes and what stays the same?" will encourage students to notice structure and there will be different ways in which the students see the structure. After some initial questions about how many popsticks would be needed to make bridges of certain lengths, students could be asked to find how many popsticks would be needed to build a bridge that contains 100 triangles as part of its Warren truss. Students should be able to demonstrate on their model how they worked it out. This is a good time for students who 'see' the problem differently to share their reasoning with other members of the class. In this way students hear of other visualisations and notice different structures. From this point students could be asked to generalise their result in words, pictures or symbols. A range of generalisations that have been observed in classrooms are included in pictorial and symbolic form in Figure 6.

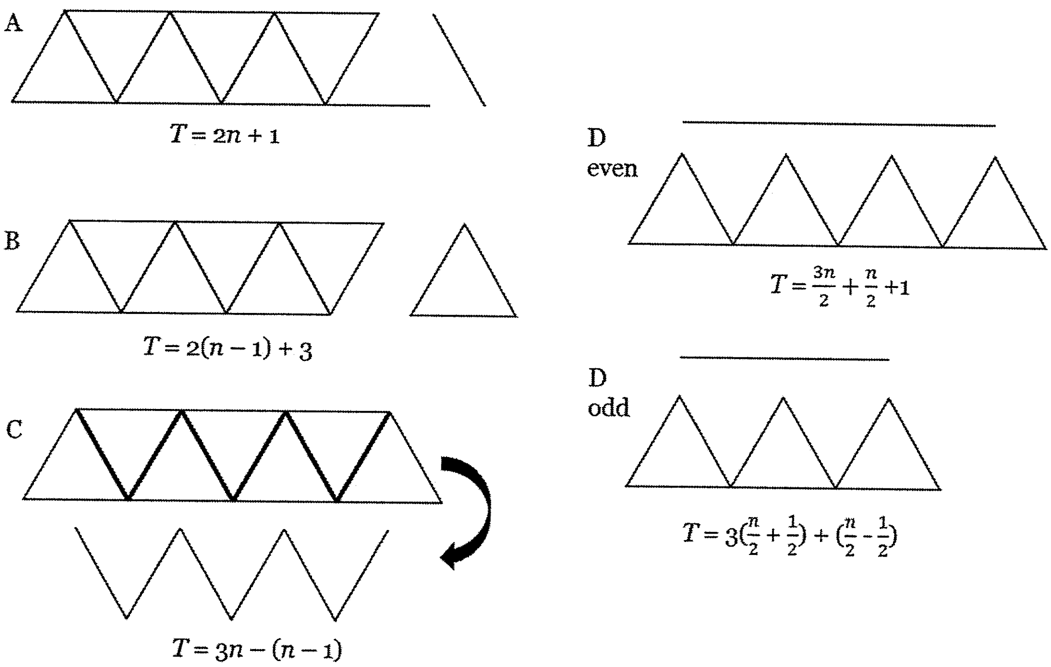

Figure 6. Warren truss bridge generalisations.

Another type of task that is suitable for students noticing structure is the investigation of number structures. For example students might investigate the sums of odd and even numbers, the property of commutativity, or the sums of consecutive numbers (Driscoll, 1999, Lovitt & Williams, 2015). One activity that assists students to recognise structure uses the story of the 18th century mathematician Carl Frederich Gauss being given the task by his teacher to add up all the numbers from one to 100. Students can use concrete materials to represent a simpler version of this problem by looking at how they might add the numbers from one to ten (see Figure 7).

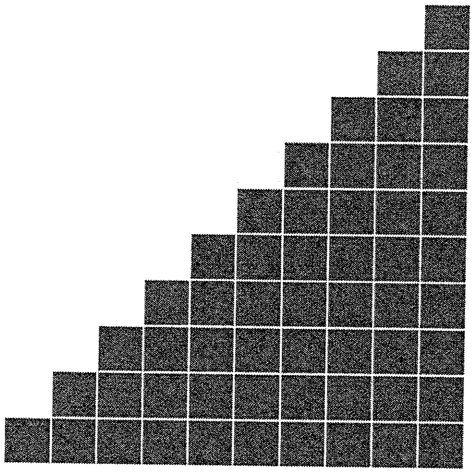

Figure 7. Concrete representation of adding the numbers from one to ten.

Through small group discussions about how these numbers could be combined in some way to make the addition easier there are generally three strategies that emerge in classrooms. These strategies represent three different visualisations that students see, and all are valid ways to solve this problem. Some students think that number combinations to ten are easy to work with, so they group their concrete materials in tens (see Figure 8). Other students notice that if they add the lowest number and the highest number and keep doing this with pairs of numbers that they have five equal groups of 11 (see Figure 9). Occasionally students will use the knowledge that arrays are useful representations and combine their model with a neighbouring group's model to form an array which is double the number that is required (see Figure 10).

When different groups share their strategies with the class it can be seen that those who visualised the problem in groups of ten found the solution $T = (5 \times 10) + 5 = 55$, those who saw the problem in groups of 11 found the solution $T = 5 \times 11 = 55$ and those who formed a rectangular array found the solution $T = (11 \times 10)/2 = 55$. Once students have identified their preferred method for summing the numbers from one to ten they can apply a similar strategy to the original challenge of summing the numbers from one to 100. These types of problems allow students to focus on the structure of the numbers, rather than just adding numbers together.

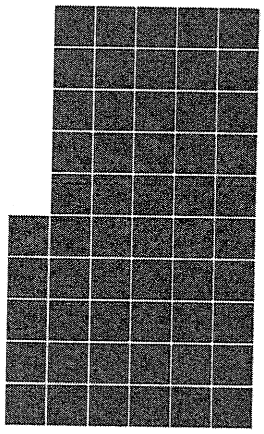

Figure 8. Grouping in tens.

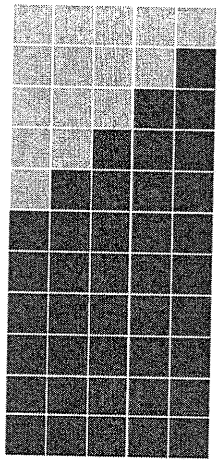

Figure 9. Grouping in 11s.

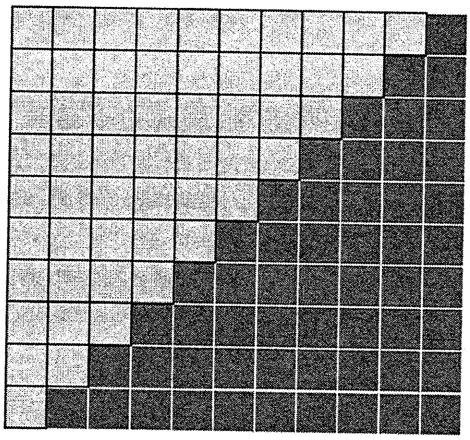

Figure 10. Forming a rectangular array.

Teachers can follow an activity like this by setting further challenges such as finding strategies for adding all the even numbers from one to 100, or all the numbers on a zero to 99 chart, or all the numbers on a multiplication chart. The major challenge could be to work out a strategy to find the total of all the numbers from one to n, the generalisation of the problem.

Conclusion

Teachers may use the development of number and algebra together as a powerful tool towards algebraic reasoning through the process of generalisation by providing students with a variety of ways in which to notice structure. Whereas arithmetic thinking tends to be about a product, finding the correct answer, algebraic thinking and reasoning is about the process of noticing pattern and structure in a variety of contexts. The noticing of structure assists students to make sense of the mathematics rather than just applying operations on numbers without necessarily understanding why they are doing so. Understanding how our number system is structured greatly helps students to reason mathematically.

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