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Developing learning progressions to support mathematical reasoning in the middle years - algebraic reasoning

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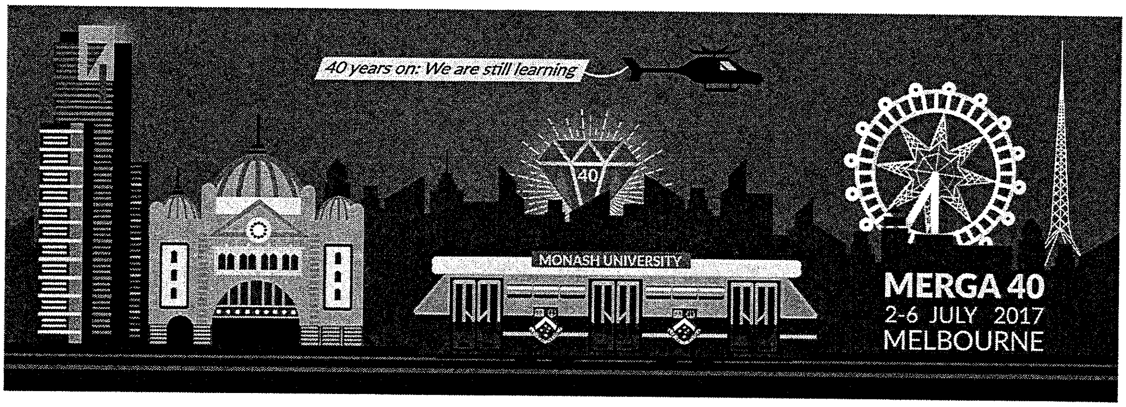

40 years on: We are still learning!

Proceedings of the 40th Annual Conference of the Mathematics Education Research Group of Australasia

Edited by Ann Downton, Sharyn Livy, & Jennifer Hall

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Preface

This publication is a record of the proceedings of the celebratory 40th conference of the Mathematics Education Research Group of Australasia (MERGA), which, like the inaugural MERGA conference, was held at Monash University in Clayton, Melbourne. The proceedings are made available to conference delegates on a USB and are also published on the MERGA website at www.merga.edu.au.

The theme of this 40th anniversary conference was *40 years on: We are still learning!* This theme was chosen to acknowledge the significant contributions of Australasian researchers over the past 40 years, was inspired by a group of currently active researchers who attended both MERGA1 and MERGA40, and is linked to the Monash University motto, *Ancora Imparo* (We are still learning). The theme also highlights the impact and importance of our collective research for enabling new learning, innovation, and critique of mathematics education for those in our region and beyond.

MERGA40 conference participants presented research papers, symposia, round table discussions, and short communications that covered a broad range of topics relevant to mathematics education across all countries, with a particular focus on the Australasian region. The MERGA40 conference also included a series of nine workshops focused on research-related issues and 15 Research Interest Area (RIA) discussion groups aligned with chapter themes in the most recent four-yearly review of mathematics education research in Australasia (Makar et al., 2016). All workshops and RIA discussion groups were led by MERGA members who are acknowledged in the proceedings and conference program. We thank these members for their important contribution, leadership, and generosity.

In accordance with established MERGA procedures, all research papers were blind peer-reviewed by panels of mathematics education researchers with appropriate expertise in the field. Papers were accepted for presentation only, or for both presentation and publication in the conference proceedings. Only those research papers accepted for presentation and publication are published in full in these proceedings. Symposia papers and the abstracts of all short communications and round tables were also peer-reviewed. The published proceedings include the keynote papers; the Beth Southwell Practical Implications Award paper; symposia papers; abstracts for round tables, short communications, and research papers accepted for presentation; and the titles of all workshops and Research Interest Area discussion groups.

We acknowledge, with gratitude, the efforts of the MERGA40 review panel chairs, reviewers, and the Monash editorial team, in reading and providing constructive feedback to presenters in a short timeframe. Ensuring that the published papers are of a high academic quality is an important responsibility of the MERGA community. We thank the proceedings editors, Ann Downton, Sharyn Livy, and Jennifer Hall, for their hard work and care in preparing these proceedings for publication.

Ann Gervasoni and Helen Forgasz
(Co-Conveners of the MERGA40 conference on behalf of the MERGA40 Monash organising committee)

Reference

Makar, K., Dole, S., Visnovska, J., Goos, M., Bennison, A., & Fry, K. (Eds.). (2016). *Research in mathematics education in Australasia 2012–2015*. Singapore: Springer.

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Reframing Mathematical Futures: Using Learning Progressions to Support Mathematical Thinking in the Middle Years

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The Australian Curriculum: Mathematics calls for the concurrent development of mathematical skills and mathematical reasoning. What are the big ideas of mathematical reasoning and is it possible to map their learning trajectories? Using rich assessment tasks designed for middle-years students of mathematics, this symposium reports on the preliminary phase of a large national study designed to move beyond the hypothetical and to provide an evidence-based foundation for learning progressions in mathematical reasoning in three key areas of the curriculum: Algebraic Reasoning, Geometrical and Spatial Reasoning, and Statistical Reasoning.

Paper 1: Dianne Siemon. *Developing Learning Progressions to Support Mathematical Reasoning in the Middle Years – Introducing the Reframing Mathematical Futures II Project*

This paper presents an overview of the project and discusses the importance of mathematical reasoning.

Paper 2: Lorraine Day, Max Stephens, & Marj Horne. *Developing Learning Progressions to Support Mathematical Reasoning in the Middle Years – Algebraic Reasoning*

The results of the initial trialling of a set of items designed to identify algebraic reasoning, and the big ideas of algebra will be discussed.

Paper 3: Marj Horne & Rebecca Seah. *Developing Learning Progressions to Support Mathematical Reasoning in the Middle Years* – Geometric Reasoning

Little recent research addresses geometrical and spatial reasoning. This paper reports on a hypothesised learning hierarchy and the results from the trial process.

Paper 4: Jane Watson & Rosemary Callingham: *Developing Learning Progressions to Support Mathematical Reasoning in the Middle Years – Statistical Reasoning*

Using an existing research base, and the outcomes from trial tests, this paper describes a learning hierarchy of statistical reasoning.

(2017). In A. Downton, S. Livy, & J. Hall (Eds.), *40 years on: We are still learning! Proceedings of the 40th Annual Conference of the Mathematics Education Research Group of Australasia* (p. 650). Melbourne: MERGA.

Developing Learning Progressions to Support Mathematical Reasoning in the Middle Years: Algebraic Reasoning

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As part of the Reframing Mathematical Futures II Project on Mathematical Reasoning, algebraic reasoning was identified as one of the three areas to be investigated. This involved developing a hypothetical learning progression for algebra to inform the design of assessment tasks to test the progression. The assessment forms were then sent to trial schools and the data was analysed using Rasch Analysis. This paper reports on the analysis of the preliminary data received and outlines some implications for teaching.

The Australian Curriculum: Mathematics (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2016) has combined Number and Algebra in a single strand to allow both to be developed together. Developing both numerical and algebraic reasoning together provides students with the opportunity to notice structure and powerful schemes for thinking about number patterns and relationships (Carpenter, Franke, & Levi, 2003). This implies that classroom practices need to adapt to build a more robust understanding of mathematics as a process of generalisation and formalisation, or as Kaput (1998) expressed it, 'algebrafying' the process. This transformation could be viewed as moving classroom practice from one of following rules and memorisation to one of sense-making (Flewelling, Kepner, & Ewing, 2007; Schoenfeld, 2008).

In order to identify a hypothetical learning progression for algebraic reasoning a review of the literature was conducted to identify the big ideas of algebra. Although the focus was to be on algebraic reasoning, it was considered appropriate to identify algebraic content, as students, at different levels, need content about which to reason. Underpinning this content focus was the understanding that in order to reason algebraically at the highest level involves visualisation, being able to move fluidly between multiple representations and having the language and discourse to reason mathematically.

Initially, hypothetical learning progressions were developed for five big ideas in algebra identified as: Pattern and Sequence, Generalisation, Function, Equivalence, and Equation Solving (Blanton, & Kaput, 2011; Blanton et al., 2015; Carraher, Schliemann, Brizuela, & Earnest, 2006; Fujii & Stephens, 2001; Mason, Stephens, & Watson, 2009; Panorkou, Maloney, & Confrey, 2013; Perso, 2003; Stephens & Armanto, 2010; Watson, 2009). However, as there was considerable overlap in the descriptors at this stage, it was decided to re-organise these in terms of: Pattern and Function, Equivalence, and Generalisation. An example of the hypothetical learning progression developed for Generalisation is shown in Table 1.

Table 1
The Hypothetical Learning Progression for Generalisation

Zone	Descriptor
1	Explain a generalisation of a simple physical situation.

(2017). In A. Downton, S. Livy, & J. Hall (Eds.), *40 years on: We are still learning! Proceedings of the 40th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 655–658). Melbourne: MERGA.

2	Explore and conjecture about patterns in the structure of number, identifying numbers that change and numbers that can vary.
3	Explain generalisations by telling stories in words, with materials and using symbols.
4	Explain generalisations using symbols and explore relationships using technology.
5	Follow, compare and explain rules for linking successive terms in a sequence or pair quantities using one or two operations.
6	Use and interpret basic algebraic conventions for representing situations involving a variable quantity.
7	Use and interpret algebraic conventions for representing generality and relationships between variables and establish equivalence using the distributive property and inverses of addition and multiplication.
8	Combine facility with symbolic representation and understanding of algebraic concepts to represent and explain mathematical situations.

Once the hypothetical learning progressions were identified on the basis of prior research, assessment tasks containing one or more items were compiled into forms that were designed to evaluate the three big ideas across Zones. Some tasks/items addressed a particular big idea while others assessed several of the big ideas in a single task. For instance, the seven-item Relational Thinking task was designed to evaluate key aspects of the hypothetical learning progressions for the two big ideas of Equivalence and Generalisation (see Table 2).

Table 2
The Relational Thinking Items and Rubrics

Item No.	Item		Rubric
1	What numbers would replace the ? to make a true number sentence (the numbers may be different). Explain your reasoning ? + 521 = 527 + ?	0	No response or irrelevant response
		1	Incorrect response but with correct reasoning based on the relationship between 521 and 527
		2	Two correct numbers given but little/no reasoning
		3	Two correct numbers given where the number on the left is 6 more than the number on the right with reasoning that reflects relationship between 521 and 527
2	Find a different pair of numbers that would make the number sentence above true	0	No response or irrelevant response
		1	A different and correct pair
3	Describe how you could find all possible pairs of numbers that would make this a true sentence.	0	No response or irrelevant response
		1	Incorrect attempt at describing based on previous answers
		2	Statement regarding difference of 6 or expression showing difference
4	What numbers would replace the ? to make a true number sentence (the numbers may be different)? ? – 521 = ? - 527	0	No response or irrelevant response
		1	Incorrect response but with correct reasoning based on the relationship between 521 and 527
		2	Two correct numbers given but little/no reasoning, may include some calculations
		3	Two correct numbers given where the number on the right is 6 more than the number on the left, with reasoning that

			reflects the relationship between 521 and 527
5	Find another set of numbers that would make the number sentence in 4 true.	0 1	No response or irrelevant response A different and correct pair
6	Describe how you could find all possible pairs of numbers that would make this a true number sentence	0 1 2	No response or irrelevant response Incorrect attempt at describing based on previous answers Statement regarding difference of 6 or expression showing the difference
7	What can you say about the relationship between *c* and *d* in this equation? $c \times 2 = d \times 14$	0	No response or irrelevant response Specific solution provided ($c = 7$ and $d = 1$) or a general statement (*c* is 7 times the number *d*) Statement correctly describes the relationship (*c* is 7 times the number *d*)

Results

Rasch analysis was used to rank student responses to the algebraic reasoning tasks and create a Draft Learning Progression for Algebra. From this it was possible to identify where different student responses to each of the Relational Thinking items were located on the progression. For instance, a score of 2 on RT1 (indicated by RT1.2 in Table 3 below) was located in Zone 3 while a score of 3 on RT1 (RT1.3) was located in Zone 6. Table 2 shows a range of responses to the RT items and their relationship to the big ideas of Equivalence (Equiv) or Generalisation (Gen).

Table 3
Results of Rasch Analysis on the Relational Thinking Items

RT1.2	RT1.3	RT2.1	RT3.1	RT3.2	RT4.2	RT4.3	RT5.1	RT6.2	RT7.1	RT7.2
Zone 3	Zone 6	Zone 4	Zone 5	Zone 6	Zone 5	Zone 7	Zone 5	Zone 7	Zone 4	Zone 6
Equiv	Gen	Equiv	Equiv	Gen	Equiv	Gen	Equiv	Gen	Equiv	Gen

The different student responses indicated by the scores for each item in Table 2 range from Zone 3 to Zone 7. Those that relate to Equivalence range from Zone 3 to Zone 5. Finding a correct pair of numbers to make a correct number sentence (RT1.2) was the easiest at Zone 3. Finding another correct pair of numbers to the same question (RT2.1) was at Zone 4. Whereas, finding two correct pairs of numbers that satisfied the subtraction number sentence (RT4.4) was scaled higher at Zone 5. Components that required students to give a general explanation of a relationship were scaled at Zone 6 or Zone 7. Generalisation items were typically more difficult than Equivalence items; and among Generalisation items, as Table 2 shows, explanations involving subtraction or difference tended to be more difficult than those involving addition relationships. This confirms research findings by Stephens and Armanto (2010), Mason et al. (2009), and Carpenter et al. (2003).

In most cases incorrect responses to items in the Relational Thinking task were located in the lower Zones of the progression. For example, giving an incorrect response to the missing numbers in item 1 was scaled at Zone 1. However, an incorrect attempt at describing the relationship between the two missing numbers based on previous answers for item 2 was at Zone 4; and an incorrect attempt at describing the relationship based on previous answers for item 6 was scaled at Zone 5. These latter two results which embody incorrect or incomplete generalisations show that, for our upper primary and junior secondary students, generalisation and explanation of algebraic thinking remains quite difficult. As the research of Kaput et al. (1998), Carraher et al. (1996), and Blanton et al. (2015) demonstrated, helping students to articulate and refine their algebraic thinking, especially their algebraic reasoning and justification, are complex and challenging tasks even for capable teachers. These abilities require constant and supportive cultivation if they are to be achieved by most students. The preliminary data presented above show that they have been achieved by some students. Expanding the range of achievement, especially with respect to the development of reasoning, remains our challenge as this project moves into its next phase.

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