2015

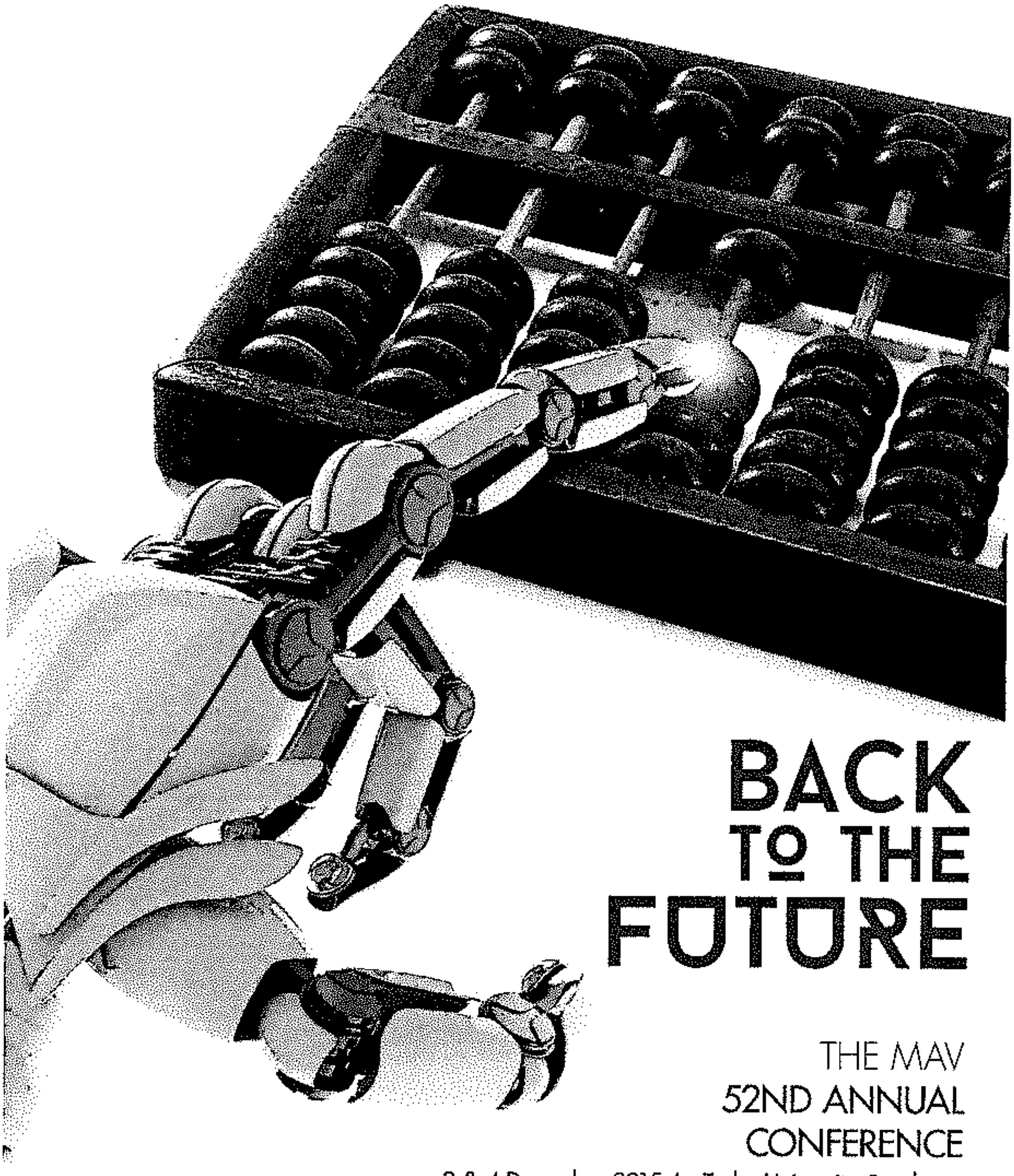
Tasks and resources for developing children's multiplicative thinking

Derek Hurrell
University of Notre Dame Australia, derek.hurrell@nd.edu.au

Chris Hurst

This conference paper was originally published as:
Hurrell, D., & Hurst, C. (2015). Tasks and resources for developing children's multiplicative thinking. *MAV Annual Conference 2015.*
http://issuu.com/julieallen35/docs/2015-mav-conference-book__1_?e=12283196/31795012

Original conference paper available here:
http://issuu.com/julieallen35/docs/2015-mav-conference-book__1_?e=12283196/31795012



BACK TO THE FUTURE

THE MAV
52ND ANNUAL
CONFERENCE

3 & 4 December 2015, La Trobe University, Bundoora

www.mav.vic.edu.au

THE MATHEMATICAL ASSOCIATION OF VICTORIA

Editorial team:
Dr Dona Martin, Ms Tina Fitzpatrick, Dr Simon Smith, Dr Christopher Lenard, Dr Sharyn Livy, Emeritus Professor Terence Mills, Dr Jonathon MacLellan, Ms Diane Itter, Dr Katherine Seaton, Ms Kym Barbary, Ms Cindy Tassone

Published December 2015 by
The Mathematical Association of Victoria
"Cliveden"
61 Blyth Street
Brunswick VIC 3056
Designed by Idaho Design & Communication

National Library of Australia Cataloguing-in-Publication Data:

ISBN: 978-1-876949-59-4

Should readers wish to contact any of the authors, full addresses are available from the Professional Officer, the Mathematical Association of Victoria.

The opinions expressed in this publication do not necessarily reflect the position or endorsement of the Editors or of the Mathematical Association of Victoria.

CONTENTS

***** Double blind reviewed paper**
**** Peer reviewed paper**
*** Summary paper**

TASKS AND RESOURCES FOR DEVELOPING CHILDREN'S MULTIPLICATIVE THINKING

Dr Derek Hurrell

University of Notre Dame Australia

Dr Chris Hurst

Curtin University

The development of multiplicative thinking determines largely the extent of the mathematics that a person learns beyond middle primary school. Our current research project has so far revealed that many primary children have a procedural view of aspects of multiplicative thinking that we believe inhibits their progress. This workshop focuses on some of the teaching resources and tasks that have been developed from our research. The purpose of these tasks is to promote the development of conceptual understanding of 'the multiplicative situation' and the many connections within it and with other big ideas such as proportional reasoning and algebraic thinking.

Multiplicative Thinking

In their discussion paper "School mathematics for the 21st century: What should school mathematics of the 21st Century be like?" (2009), the peak body for mathematics education in Australia, the Australian Association of Mathematics Teachers (AAMT) stated: "The so-called Big Ideas in Mathematics are key to connecting other aspects of mathematics, both between and within the Mathematical Concepts and Mathematical Actions. They are overarching ideas that are neither 'concepts' nor 'actions'. Most of the "Big Ideas" pervade a number of conceptual areas of mathematics, and provide connections between them" (p.4). Basically the big ideas are those central ideas fundamental to mathematical success. According to research (AAMT, 2015; Charles, 2005; Siemon, Bleckly and Neal, 2012), one of the big ideas is multiplicative thinking. It is therefore imperative that for continued success students become 'solid' multiplicative thinkers (Siemon, Breed, Dole, Izard, & Virgona, 2006).

Multiplicative thinking is vitally important in the development of significant mathematical concepts and understandings such as algebraic reasoning (Brown & Quinn, 2006), place value, proportional reasoning, rates and ratios, measurement, and statistical sampling (Mulligan & Watson, 1998; Siemon, Izard, Breed & Virgona, 2006). Further, Siegler et al. (2012) advocate that knowledge of division and of fractions (another part of mathematics very much reliant on multiplicative thinking) are unique predictors of later mathematical achievement.

The Difficulties of Multiplicative Thinking

Unfortunately, research (Clark & Kamii, 1996; Siemon, Breed, Dole, Izard, & Virgona, 2006) has found that the label of being 'solid' multiplicative thinkers cannot be applied to most students. Clarke and Kamii (1996) found that 52% of fifth grade students were not 'solid' multiplicative thinkers, and Siemon, Breed, Dole, Izard, and Virgona (2006) established that up to 40% of Year 7 and 8 students performed below curriculum expectations in multiplicative thinking, with at least 25% well below the expected level.

Whereas most students enter school with informal knowledge that supports both counting and early additive thinking (Sophian & Madrid, 2003) students need to re-conceptualise their understanding about number to understand multiplicative relationships (Wright, 2011). Multiplicative thinking is distinctly different from additive thinking even though it is constructed by children from their additive thinking processes (Clark & Kamii, 1996). The difference between additive thinking and multiplicative thinking has been characterised by Confrey and Smith (1995) as the difference between a "counting world" and a "Splitting world". Essentially a "splitting world" is the ability to share (split) and is an idea to which many students are very sensitive from their earliest experiences (Confrey, 1994). Therefore, this makes splitting part of the multiplicative situation. The counting world however, identifies additive increments and often 'interferes' with the splitting concept. The counting world does not lead students' thinking into the world of rational numbers (fractions, percentages ratios etc.) in the same way as the Splitting world does (Confrey & Smith, 1995).

Multiplicative Thinking. What Helps Students?

If, as the research tells us, multiplicative thinking is vital for further success in mathematics, but difficult to learn, then teachers need the content and pedagogical knowledge to succeed in their endeavours to effectively teach it. Carroll (2007) has constructed a list of strategies and ideas that support multiplicative thinking.

- Allow children to work out their own ways to solve problems involving multiplicative thinking.
- Compare additive and multiplicative thinking approaches.
- Use models that clearly illustrate the idea/s.
- Sometimes students are introduced to the ideas symbolically before the groundwork has been done to establish meaning and become comfortable in working with them.
- Make and discuss the links between fraction ideas, rates, ratios and proportion.
- Use authentic contexts and models to exemplify situations.
- Estimation is really important as it demonstrates understanding of the concepts involved.
- Engage in conversations about the ideas and talk about the links, discuss the similarities and differences between the ideas.
- It is development of fuller, deeper and more connected understandings of the number system that makes a difference.

Carroll (2007, pp. 41 - 42)

In the remainder of this article we would like to pursue dot point three of Carroll's (2007) list; "Use models that clearly illustrate the idea/s". Although this will be the focus, it should be noted that by carefully considering the model used to 'carry' the understanding, at one time or another, all of the other dot points could and definitely should be exercised.

A Model for the Development of Multiplicative Thinking

One model for trying to build a conceptual understanding of multiplication is the multiplicative array. Multiplicative arrays refer to representations of rectangular arrays in which the multiplier and the multiplicand are exchangeable (Figure 1). These arrays are seen by some as powerful ways in which to represent multiplication (Barmby, Harries, Higgins & Suggate, 2009; Young-Loveridge & Mills, 2009). Young-Loveridge (2005) asserts that multiplicative arrays have the potential to allow students to visualise commutativity, associativity and distributivity. Further, Wright (2011) states that multiplicative arrays embody the binary nature of multiplication, and that as a representation, they have value, as they also connect to ideas of measurement of area and volume and Cartesian products.

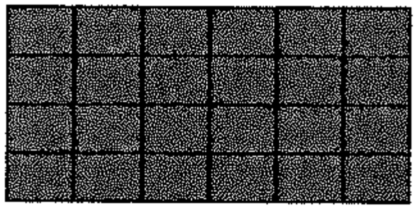


Figure 1: Multiplicative array

We will visit the use of multiplicative arrays in the activity called "A bag of tiles."

Task - A bag of Tiles

This task can be varied to suit the teaching and learning of several aspects of multiplicative thinking. Essentially it is based on students working with a set of 2cm x 2cm plastic tiles. These can be given out in a plastic snap-lock bag and can vary in number, depending on the task and the targeted aspect of multiplicative thinking. However for the following activity each pair of students is given 24 tiles. Although each student could be given their own set of tiles, if we want the students to engage in meaningful conversations about the task and about the multiplicative thinking behind the task, then having the students in pairs is actually a more productive setting.

The basic task is for children to make an array with the tiles so that rows and columns contain the same number of tiles with none left over. In this case it is an array of 24 tiles in a 6 x 4 configuration. The different arrays A and B (Figure 2) provide an interesting discussion point in leading children to a realisation that although the two arrays arrive at the same total, the manner in which they are constructed is important in certain contexts. For instance, there may be very big ramifications in not understanding that, although in a day you would end up taking twelve tablets, taking two tablets, six times a day may have very different effects from taking six tablets twice a day. This idea can be further developed by asking children to 'tell a story' about each number fact to show that 4 x 6 (four rows of six) is indeed different to 6 x 4 (six rows of four).

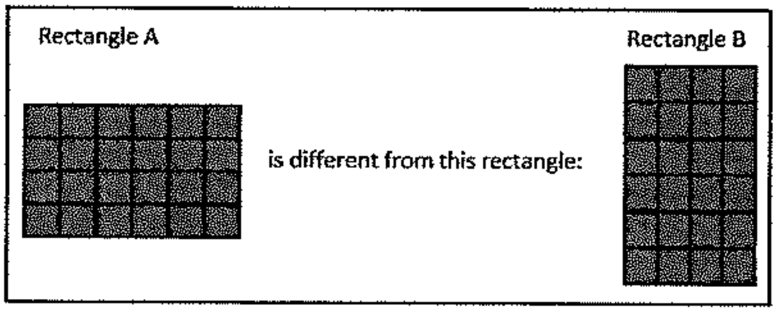


Figure 2: Two "different" rectangles, 6 x 4 and 4 x 6

As well, the two arrays offer a good opportunity to develop an understanding of the commutative property in a deeper way that by simply rotating the array. Different coloured strips of four and six squares can be laid over the array to show that while the two arrays represent the same product, they are in fact different (Figure 3).

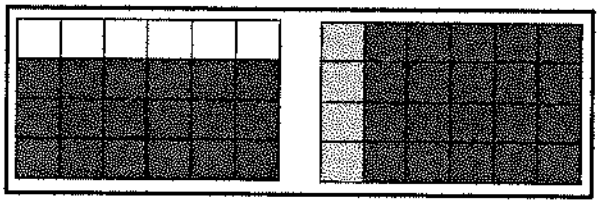


Figure 3: Overlaying Rectangle A with different strips

The students are then asked to find all of the different rectangles that can be made from 24 tiles. The bag of tiles activity works very well as a physical representation to develop an understanding of factors and multiples as well as the commutative property of multiplication, both very powerful understandings which will be often called upon in mathematics. This is also a good opportunity to make connections between the representations of the arrays and the way we symbolically record them. Further we can exploit the opportunity to discuss and show the links between the multiplicative situations of multiplication and division, that is; 6 x 4 = 24, 4 x 6 = 24, 24 ÷ 6 = 4 and 24 ÷ 4 = 6. Over and above the mathematical content that this activity contains, it also embodies the four proficiency strands (Problem Solving, Understanding, Reasoning and Fluency) as articulated in the Australian Curriculum: Mathematics (ACARA, 2015).

Further to the richness that can be gleaned from using 24 tiles and asking the questions that are articulated above, the same activities can be entered into with arrays of other sizes to further develop and re-inforce the understandings. The tiles can then be employed to investigate prime, composite, square and triangular numbers.

For this activity the students are given more tiles to work with, and are instructed that they cannot rotate the tiles (employ the commutative property) and still consider them to be 'different'. Therefore a 6 x 4 configuration is considered to be the same as a 4 x 6 configuration. They are then asked see if they can make a rectangle with two tiles in more than one 'different' way? They build the array and record the finding that the only configuration for two tiles is a 2 x 1 array. The students then investigate three tiles and continue their investigations as required, but probably for not less than the 24 tiles with which they started (Figure 4). There is a conversation to be had with first of all, four tiles, and then with nine tiles, and then 16 tiles (possibly even 25 tiles) about whether a square is a rectangle.

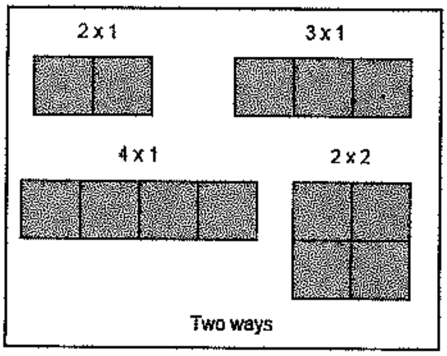


Figure 4: Arrays for 2 tiles, three tiles and four tiles

Initially what the students are making and recording are the factors for whole numbers between two and 24. What is also occurring, is an opportunity to talk about the numbers for which two or more sets of factors cannot be found (prime numbers) and the numbers which have multiple sets of factors (composite numbers). The exploration here of course is what makes some numbers prime numbers and others, composite numbers. Also, by previously considering arrays for the numbers four, nine, 16 and perhaps 25, an exploration of square numbers and why we call them square numbers can be undertaken.

Conclusion

In this article we have only just begun to scratch the surface of the opportunities afforded by multiplicative arrays to support students in understanding the multiplicative situation. Arrays can also be used to link to the division construct for fractions – e.g., a group of twenty four tiles can be split into quarters so that one quarter of 24 is 6, two quarters of 24 is 12, three quarters of 24 is 18. The same can be said about sixths. Further the concept of why we calculate area as we do can be explored by overlaying the tile array with a clear grid of the same number of squares (i.e., for a 4 x 6 array, use a clear grid of 2 cm squares in a 4 x 6 pattern). Students are asked to describe the area of the grid.

What the array offers is a powerful illustration of multiplicative situations and rich opportunities to use this construct to problem solve, develop understanding, practice fluency and cultivate reasoning.

References

Australian Association of Mathematics Teachers (AAMT) (2015). *Desktop Review of Mathematics School Education: Pedagogical Approaches and Learning Resources.* Retrieved from http://www.aamt.edu.au/Library/Other

Australian Association of Mathematics Teachers (AAMT) (2009). *Discussion paper: School mathematics for the 21st century.* Retrieved from http://www.aamt.edu.au/Library/Other/Maths-for-the-21st-century

Australian Curriculum, Assessment and Reporting Authority (ACARA) (2015). *Australian curriculum mathematics – Version 7.5.* Retrieved from http://www.australiancurriculum.edu.au/Mathematics/Curriculum/F-10.

Barmby, P., Harries, T., Higgins, S., & Suggate, J. (2009). The array representation and primary children's understanding and reasoning in multiplication. *Educational Studies in Mathematics, 70,* 217–241.

Carroll, J. (2007). *Developing Multiplicative Thinking.* Retrieved from http://makingmathsmarvellous.wikispaces.com/file/view/MultiplicativeThinking.pdf

Charles, R.I. (2005). Big ideas and understandings as the foundation for early and middle school mathematics. *NCSM Journal of Educational Leadership, 8*(1), 9–24.

Clark, F. B., & Kamii, C. (1996). Identification of multiplicative thinking in children in grades 1-5. *Journal for Research in Mathematics Education, 27,* 41-51.

Confrey, J. (1994). Splitting, similarity, and rate of change: New approaches to multiplication and exponential functions. In G. Harel & J. Confrey (Eds.), *The Development of Multiplicative Reasoning in the Learning of Mathematics* (pp. 293-332). Albany, NY: State University of New York Press.

Confrey, J. & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education, 26*(1), 66-86.

Siegler R.S., Duncan G.J., Davis-Kean P.E., Duckworth K., Claessens A., Engel M., Susperreguy M.I., & Chen M. (2012). Early Predictors of High School Mathematics Achievement. *Psychological Science, 23* (7), 691-697.

Siemon, D., Bleckly, J. and Neal, D. (2012). Working with the Big Ideas in Number and the Australian Curriculum: Mathematics. In B. Atweh, M. Goos, R. Jorgensen & D. Siemon, (Eds.). (2012). *Engaging the Australian National Curriculum: Mathematics – Perspectives from the Field.* Online Publication: Mathematics Education Research Group of Australasia pp. 19-45.

Siemon, D., Breed, M., Dole, S., Izard, J., & Virgona, J. (2006). *Scaffolding Numeracy in the Middle Years – Project Findings, Materials, and Resources,* Final Report submitted to Victorian Department of Education and Training and the Tasmanian Department of Education, Retrieved from http://slideplayer.com/slide/231010/

Siemon, D., Izard, J., Breed, M., & Virgona, J. (2006). *The derivation of a learning assessment framework for multiplicative thinking.* Paper presented at the 30th Conference of the International Group for the Psychology of Mathematics Education – Mathematics in the Centre, Prague.

Sophian, C., & Madrid, S. (2003). Young childrens' reasoning about many-to-one correspondences. *Child Development, 74*(5), 1418-1432.

Wright, V. J. (2011). *The development of multiplicative thinking and proportional reasoning: Models of conceptual learning and transfer.* (Doctoral dissertation). University of Waikato, Waikato. Retrieved from http://researchcommons.waikato.ac.nz/.

Young-Loveridge, J. (2005). Fostering multiplicative thinking using array-based materials. *Australian Mathematics Teacher, 61*(3), 34-40.

Young-Loveridge, J., & Mills, J. (2009). *Teaching multi-digit multiplication using array-based materials.* In R. Hunter, B. Bicknell, & T.Burgess (Eds.), Crossing divides (Proceedings of the 32nd annual conference of the Mathematics Education Research Group of Australasia pp. 635-643). Palmerston North, NZ: MERGA

INVESTIGATING CHILDREN'S MULTIPLICATIVE THINKING

Dr Chris Hurst

Curtin University

Dr Derek Hurrell

University of Notre Dame Australia

Multiplicative thinking is a 'big idea' of mathematics that underpins much of the mathematics learned beyond the early primary school years. The conference presentation reports on a recent study that utilised an interview tool to gather data about children's multiplicative thinking. Using a workshop format, we present some of the interview tool and some of the findings, as well as demonstrate how the tool can be used for planning, teaching and assessment. The session also emphasises the importance of developing deep conceptual understanding as opposed to the teaching of procedures. This paper considers how evidence from the interview can be used to inform teaching.

Multiplicative Thinking

The importance of multiplicative thinking as a 'big idea' of mathematics has been well documented (Siemon, Bleckly, & Neal, 2012; Siemon, Breed, Dole, Izard, & Virgona, 2006), as has the importance of 'big ideas' in highlighting the myriad connections within and between them (Charles, 2005; Clarke, Clarke, & Sullivan, 2012). Charles (2005) asserted that 'big ideas' "link numerous mathematical understandings into a coherent whole", make connections, and that "good teaching should make those connections explicit" (p. 10). Multiplicative thinking is one such 'big idea'.

It therefore seems to follow, that to make these explicit links to develop multiplicative thinking, teachers should incorporate the Proficiency Strands (particularly Reasoning and Problem Solving) of the Australian Curriculum: Mathematics (Australian Curriculum, Assessment & Reporting Authority, 2015) rather than focus solely on the Content Strands (Number and Algebra, Measurement and Geometry, Statistics and Probability). For example, instead of teaching children a set of 'rules' for working with numbers, and teaching ideas like multiplication and division as separate entities, more effective teaching would focus on reasoning about why numbers behave as they do when operating, and understanding how multiplication and division are different ways of describing the same situation. This paper describes some research conducted with primary aged children to determine the extent of their multiplicative thinking. The results of that are very interesting in themselves. However, it is the inferences that can be drawn about teaching and the associated implications for teaching about multiplicative thinking that comprise the main thrust of this paper.

The Research

Semi-structured interviews were conducted with thirty eight children in Years Five and Six in two different schools (Schools A and B). Interviews lasted between 25 and 40 minutes. A questionnaire was developed from the interview format in order to generate a larger set of data in a shorter time. This was administered to nine whole class groups comprising 180 children in Years Four, Five and Six at a third school (School C) and the administration of the questionnaire took about 30 minutes per group. Both the questionnaire and interview were administered to the Year Five group at School A to establish the reliability of the results from the questionnaire. Whilst the data from the questionnaire was shown to be reliable in that the results from it were reflected in those from the interview, richer data were generated from the interview. Burns (2010) asserts that the power of the interview lies in the quality of the question posed by the interviewer or teacher. Examples include "Can you explain how you worked that out?" and "How did you get that answer?" irrespective of whether the child interviewee had the correct answer or not.

Typical questions from the interview and questionnaire included the following:

- In $7 \times 6 =$, what do the numbers 7 and 6 tell you?
- Do a drawing to show the number fact (or table) 4×3.

- Write as many multiplication facts (or tables) as you can that give an answer of 24. Circle all of the numbers that are factors and draw a square around numbers that are multiples. Explain what they are factors and multiples of, and how you know.
- My friend says that if you know the answer to 6×17, you must also know the answer to 17×6. Is he correct? Why/why not?
- My friend says that if you know that $6 \times 17 = 102$, you must also know the answer to $102 \div 6$? Is he correct? Why/why not?

These questions are chosen because they explore key aspects of multiplicative thinking, the understanding (or otherwise) of which is likely to provide an indication of a student's level of thinking.

Overview of Results

The responses from the interviews and questionnaires made for some interesting overall observations. First, responses from the Year Five cohort at School A and the Year Six cohort at School B revealed a wide range of conceptual understanding. Second, responses to the questionnaire administered at School C revealed that the three class groups within each year level had varying levels of understanding. While there were large variations within each year level, a similar range was evident between year levels, and indeed, within each class group. This paper suggests that the differences may have resulted, at least to some extent, from different pedagogies, teaching styles, and/or may reflect different stages of development of children's understandings of multiplicative concepts. After all, Siemon et al. (2011) have noted that multiplicative thinking usually does not fully develop until the early secondary years.

Specific Results and Discussion

School A and School B

The purpose of this paper is not to compare performance of different school cohorts against one another or different sections of school cohorts against one another. Rather it seeks to identify aspects of multiplicative thinking that might be evident or otherwise in different children and to understand why that might be so. Hence interview results from Schools A and B are combined into one set. Table 1 presents a summary of responses to the five illustrative questions listed above for the Year Five cohort from School A and the Year Six cohort from School B (n = 38).

Table 1 - *Summary of Responses from Schools A and B*

Mathematical understanding demonstrated by responses to listed questions	School A Year Five School B Year Six (n=38)
Identifies numbers in multiplication fact as 'group size' and 'number of groups'.	39%
Represents given multiplication fact as an array.	34%
Defines 'factor' and 'multiple' and/or identifies factors and multiples in given number fact.	63%
Explains commutative property in a conceptual way and/or demonstrates it using an array.	29%
Explains inverse relationship in a conceptual way based on number of groups and group size	50%

It is also worth noting that of the 38 children in the combined sample, ten (26%) responded correctly to four or five of the above questions and a further seven (18%) responded correctly to three of the questions. This seems to indicate that approximately one quarter of the sample demonstrated a strong level of conceptual understanding of the selected aspects of multiplicative thinking and a smaller proportion showed a reasonable level of understanding. However, over half the children in the sample could only respond appropriately to two or less of the selected questions. This suggests that there is a wide range of understanding across the sample.

Some Typical Strong Responses

Typical responses demonstrating a strong level of conceptual understanding of the commutative property of multiplication include the following:

- Student Dylan – "It doesn't really matter which way it is – seventeen groups of six is the same as six groups of seventeen". He then used tiles to make three groups of five and five groups of three, and also rearranged twelve tiles saying "I just put them into a three by four grid – it's the same as a four by three".
- Student Dean – "It's just the same . . . you just flip it around". He then used tiles to make a three by five array and rotated the array to explain his point.
- Similarly, the following exchange during the interview with Student Jason shows some connection of ideas around the inverse relationship between multiplication and division, sharing into equal groups, and arrays.
- When discussing the division fact 24 ÷ 3, Jason showed it as an array and said, "Then I'm going to split it up into threes, because I'm going to see how many groups of three

I can have in 24". Also said, when asked what the answer would be, "I started with knowing that how many threes go into 15 and that's five, then I counted by threes to get 18, 21, 24". He also said, "If I had 3 times 4 it would be 12. If I had 12 divided by 3 it would be 4". He also gave a similar example with "8 groups of 3 = 24, So 24÷8 = 3"

Such connected discussion seems to demonstrate a sound understanding of the concepts involved.

Responses Indicating Partial Understanding

The apparent lack of conceptual understanding in the responses of some children is of interest. It is difficult to draw conclusions about the depth of some children's conceptual understanding given the absence of links and connections between responses to different questions. That is, some children show some understanding of a particular idea which would lead one to reasonably expect they would show an understanding of related concepts. However, this was often not the case.

It is well accepted that the array is a powerful representation of the multiplicative situation (Jacob & Mulligan, 2014). However, while two of the children (Ellie and Tilly) drew an array to represent the given number fact, neither of them could explain why the commutative property works, in terms of the array. Rather, they said that the numbers were 'swapped around' (Tilly) or 'you've just swapped them around' (Ellie). Also, of the thirteen children who drew an array, only seven of them could describe factors and multiples.

Similarly, while 63% of the children (n = 24) could adequately describe factors and multiples and their roles in the multiplicative situation, only five of them talked about the 7×6 number fact in terms of group size and number of groups. Further to that, 39% of the children (n = 15) described the number fact in terms of group size and number of groups, yet only five of that group also drew an array. Some of the children (n = 11) adequately explained the commutative property and half (n = 19) explained the inverse relationship between multiplication and division. However, not all of the eleven children who explained the commutative property could also explain the inverse relationship. This is interesting because the ideas underpinning those interview tasks are inextricably linked – that is, group size/number, the factor-factor-multiple relationship, the representation as an array, the commutative property, and the inverse relationship. Hence, it might be reasonably expected that there would be more children who could perform well on all or most of the items, or on none (or very few) of them.

School A and B Implications

The inferences that can be drawn from the School A and B data suggest that the connections between those important ideas need to be made clear and more explicit so that a mutual understanding of them can be developed. This is supported by the observations that: some students drew an array; some others could explain factors and multiples; and some others could explain a multiplication fact in terms of group size and number. Perhaps it is because there has been passing mention made of these key ideas, rather than sustained and explicit teaching of them. For example, the fact that less than one third of the children could explain the commutative property in a conceptual way gives rise to questions about how the commutative property may have been taught. Perhaps it is also attributable to the fact that children's understanding of the multiplicative situation is developing and in a state of flux. After all, it has been noted (Siemon et al., 2011) that multiplicative thinking is a concept that does not fully develop until the secondary years around the age of fourteen and the students involved here are several years younger than that.

The lack of sustained teaching may also be because the teachers simply do not appreciate the critical importance of the ideas of factor/multiple, group size/number, and the use of the array. Hence they may have taught some of the ideas once, assuming that such exposure is adequate when that is clearly not the case.

School C

In School C, the questionnaire was administered to 180 children in Years Four, Five and Six. Table 2 represents the responses from the three year levels in School C to questions about the same concepts as shown in Table 1.

Table 2 - *Comparison of Responses from Different Year Levels at School C*

Mathematical understanding demonstrated by responses to listed questions	Year 4	Year 5	Year 6
Identifies numbers in multiplication fact as 'group size' and 'number of groups'.	14%	9%	23%
Represents given multiplication fact as an array.	28%	32%	39%
Defines 'factor' and 'multiple' and/or identifies factors and multiples in given number fact.	3%	15%	14%
Explains commutative property in a conceptual way and/or demonstrates it using an array.	0%	2%	5%
Explains inverse relationship in a conceptual way based on number of groups and group size	10%	1%	12%

In general it would probably be expected that the Year Six children would perform better than the Year Five children who would in turn perform better than the Year Four children. However, as can be seen, this is not always the case and even where it is, one would perhaps expect the comparative performance of the older children to be markedly better than it is.

Of more interest is the comparison within each year level in School C, as shown in Table 3 which shows responses from children in the three Year Four classes. Here, there are marked differences in the responses from different class groups, particularly in relation to the first two questions. It is indeed surprising that no child in Class 4A could identify 'group size' and 'number of groups' in multiplication facts when over a third (35%) of Class 4C could do so. Even more intriguing is that nearly two thirds (62%) of Class 4A drew an array to show a multiplication fact when not one child in Class 4C did that. As well, very few children in Class 4B responded correctly on any of the five questions. What does this indicate?

Table 3 - Comparison within Year Levels in School C

Mathematical understanding demonstrated by responses to listed questions	4A	4B	4C
Identifies numbers in multiplication fact as 'group size' and 'number of groups'.	0%	6%	35%
Represents given multiplication fact as an array.	62%	17%	0%
Defines 'factor' and 'multiple' and/or identifies factors and multiples in given number fact.	0%	6%	5%
Explains commutative property in a conceptual way and/or demonstrates it using an array.	0%	0%	0%
Explains inverse relationship in a conceptual way based on number of groups and group size	14%	0%	15%

School C Implications

Classes at School C are not streamed on ability. Hence, it seems reasonable to assume that the variation in responses may be due to different teaching occurring among the three Year Four classes. Perhaps there has been a clear emphasis in Class 4A on the use of arrays, rather than showing multiplication facts as separate groups. It is also worth noting that the responses from Class 4A (62%) to the array question were the highest of any class in the school – only one Year Six class (53%) and one Year Five class (50%) recorded a similar level of correct responses. Similarly, the teaching in Class 4C is likely to have emphasized the notion of 'group size' and 'number of groups' in the multiplicative situation. Again, Class 4C's response (35%) is the highest recorded of all classes with only one Year Six class (33%) recording a similar level of correct answers. However, it seems reasonable to imply that there is a need for explicit teaching of the connections between the five related ideas in the multiplicative situation, something which seems to be reflected in the responses from Schools A and B as well.

General Implications

The five selected questions from the interview and questionnaire represent less than a quarter of the full instrument yet the data generated from just three sets of children have provided plenty of food for thought. There are two main observations that can be made from the presented data. First, there are considerable differences in the levels of understanding of multiplicative concepts shown by two groups (Schools A and B) that were interviewed. Some children displayed more connected understanding than did others. Second, there is considerable difference in responses among classes in the same year level at the same school (School C) where the questionnaire was administered. In seeking reasons for this, it is reasonable to infer that the differences may be due to pedagogies.

The differences in responses are quite stark at times and the relative connectedness in the thinking of some children in the combined cohort from Schools A and B suggests that connections between ideas may have been made more explicit in some classes compared to others. At least, it is likely that some children from the School A/B group have been encouraged to justify, explain, and reason about their ideas, as well as interpret those of others. Apart from the differences in responses, some children were more forthcoming and articulate which suggests that they may have been more accustomed to discussing mathematical ideas than other children from the same combined group who were often unable to elaborate their answers. Also, in responding to questions other than those reported here, some children were reluctant and/or unable to depart from quite procedural responses which was not the case with other children.

Conclusions

In conclusion, there are implications for teaching in terms of what can be done to help children develop key multiplicative concepts in a connected way. The evidence presented here suggests that such pedagogical practices exist but may not be sufficiently widespread. Such teaching could include the following:

- Explicitly teach that the multiplicative situation is based on the number of equal groups and the size of each group.
- Develop an understanding of the terms factor and multiple through the use of arrays, and explicitly use them as 'mathematical language'.
- Teach multiplication and division simultaneously, not separately.
- Develop the commutative property through the use of arrays and physically show the 'x' rows of 'y' gives the same result as 'y' rows of 'x'.

If teachers view multiplication and division as different ways of representing 'the multiplicative situation', rather than as separate entities, the links and connections between the ideas discussed in this paper can be made explicit for children. When those connections are clearly understood, ideas such as the inverse relationship and the commutative property become much easier to grasp.

References

Australian Curriculum, Assessment and Reporting Authority (ACARA) (2015). *Australian curriculum mathematics*. Retrieved from http://www.australiancurriculum.edu.au/Mathematics/Curriculum/F-10.

Burns, M. (2010). Snapshots of students' misunderstandings. *Educational Leadership, February, 2010,* 18-22.

Charles, R.I. (2005). Big ideas and understandings as the foundation for early and middle school mathematics. *NCSM Journal of Educational Leadership, 8*(1), 9–24.

Clarke, D.M., Clarke, D.J. and Sullivan, P. (2012). Important ideas in mathematics: What are they and where do you get them? *Australian Primary Mathematics Classroom, 17*(3), 13-18.

Jacob, L., & Mulligan, J. (2014). Using arrays to build towards multiplicative thinking in the early years. *Australian Primary Mathematics Classroom, 19*(1), 35-40.

Siemon, D., Beswick, K., Brady, K., Clark, J., Faragher, R., & Warren, E. (2011). *Teaching mathematics: Foundations to middle years.* South Melbourne: Oxford University Press.

Siemon, D., Bleckly, J. and Neal, D. (2012). Working with the Big Ideas in Number and the Australian Curriculum: Mathematics. In B. Atweh, M. Goos, R. Jorgensen & D.

Siemon, (Eds.). (2012). *Engaging the Australian National Curriculum: Mathematics – Perspectives from the Field.* Online Publication: Mathematics Education Research Group of Australasia pp. 19-45.

Siemon, D., Breed, M., Dole, S., Izard, J., & Virgona, J. (2006). *Scaffolding Numeracy in the Middle Years – Project Findings, Materials, and Resources,* Final Report submitted to Victorian Department of Education and Training and the Tasmanian Department of Education, Retrieved from http://www.eduweb.vic.gov.au/edulibrary/public/teachlearn/student/snmy.ppt