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Investigating children's multiplicative thinking

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BACK TO THE FUTURE

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Back to the Future

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FOREWORD

The 52nd Annual Conference Proceedings of the Mathematical Association of Victoria contains a wonderful collection of articles on a wide selection of mathematical topics. The works, by teachers, teacher educators and researchers, offer a clear indicator that continued innovation is taking place in mathematics classes in our schools and universities. Back to the Future offers readers many thought provoking ideas through a breadth of engaging articles.

The editorial team is representative of the wide variety of MAV members. We enjoyed working together as much as we did with the authors, whom we thank individually for their dedicated commitment to progressing mathematics education.

We thank the MAV conference organisation team for their professionalism and encourage each of you to continue in your support of their most important work. The professional development focus and opportunities offered by MAV remain as valuable as ever in progressing mathematics education.

The Review Process for the Mathematical Association of Victoria 52nd Annual Conference Proceedings

Papers were submitted for double-blind review, peer review or as summaries. The Editors received 11 full papers for the double blind review process, for which the identities of author and reviewer were concealed from each other. Details in the papers that identified the authors were removed to protect the review process from any potential bias, and the reviewers' reports were anonymous. Two reviewers reviewed each of the 11 blind review papers and if they had a differing outcome a third reviewer was required. Ten of the 11 papers were accepted for publication. In addition, we received 14 full papers for the peer review process, where the names of the authors were identified to reviewers; 11 were accepted for publication as peer-reviewed papers and two were accepted as summary papers. Eight papers submitted as summary papers were reviewed by a combination of external reviewers and the editorial team. Seven of these were accepted for publication (one as a peer reviewed paper).

In the Conference Proceedings, double-blind and peer reviewed papers are grouped together and arranged in alphabetical order of author names. Double-blind reviewed papers and peer reviewed papers are indicated by ** and * respectively following the paper title. Summary papers follow the double blind and peer reviewed papers.

Of the total of 34 papers received, 33 papers are published: ten double-blind reviewed papers, seventeen peer reviewed papers and six summary papers. Altogether, 21 reviewers assisted in the review process, all of whom provided thoughtful feedback and were outstanding in responding quickly to our invitations.

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***** Double blind reviewed paper**
**** Peer reviewed paper**
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INVESTIGATING CHILDREN'S MULTIPLICATIVE THINKING

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Multiplicative thinking is a 'big idea' of mathematics that underpins much of the mathematics learned beyond the early primary school years. The conference presentation reports on a recent study that utilised an interview tool to gather data about children's multiplicative thinking. Using a workshop format, we present some of the interview tool and some of the findings, as well as demonstrate how the tool can be used for planning, teaching and assessment. The session also emphasises the importance of developing deep conceptual understanding as opposed to the teaching of procedures. This paper considers how evidence from the interview can be used to inform teaching.

Multiplicative Thinking

The importance of multiplicative thinking as a 'big idea' of mathematics has been well documented (Siemon, Bleckly, & Neal, 2012; Siemon, Breed, Dole, Izard, & Virgona, 2006), as has the importance of 'big ideas' in highlighting the myriad connections within and between them (Charles, 2005; Clarke, Clarke, & Sullivan, 2012). Charles (2005) asserted that 'big ideas' "link numerous mathematical understandings into a coherent whole", make connections, and that "good teaching should make those connections explicit" (p. 10). Multiplicative thinking is one such 'big idea'.

It therefore seems to follow, that to make these explicit links to develop multiplicative thinking, teachers should incorporate the Proficiency Strands (particularly Reasoning and Problem Solving) of the Australian Curriculum: Mathematics (Australian Curriculum, Assessment & Reporting Authority, 2015) rather than focus solely on the Content Strands (Number and Algebra, Measurement and Geometry, Statistics and Probability). For example, instead of teaching children a set of 'rules' for working with numbers, and teaching ideas like multiplication and division as separate entities, more effective teaching would focus on reasoning about why numbers behave as they do when operating, and understanding how multiplication and division are different ways of describing the same situation. This paper describes some research conducted with primary aged children to determine the extent of their multiplicative thinking. The results of that are very interesting in themselves. However, it is the inferences that can be drawn about teaching and the associated implications for teaching about multiplicative thinking that comprise the main thrust of this paper.

The Research

Semi-structured interviews were conducted with thirty eight children in Years Five and Six in two different schools (Schools A and B). Interviews lasted between 25 and 40 minutes. A questionnaire was developed from the interview format in order to generate a larger set of data in a shorter time. This was administered to nine whole class groups comprising 180 children in Years Four, Five and Six at a third school (School C) and the administration of the questionnaire took about 30 minutes per group. Both the questionnaire and interview were administered to the Year Five group at School A to establish the reliability of the results from the questionnaire. Whilst the data from the questionnaire was shown to be reliable in that the results from it were reflected in those from the interview, richer data were generated from the interview. Burns (2010) asserts that the power of the interview lies in the quality of the question posed by the interviewer or teacher. Examples include "Can you explain how you worked that out?" and "How did you get that answer?" irrespective of whether the child interviewee had the correct answer or not.

Typical questions from the interview and questionnaire included the following:

- In $7 \times 6 =$, what do the numbers 7 and 6 tell you?
- Do a drawing to show the number fact (or table) 4×3.

- Write as many multiplication facts (or tables) as you can that give an answer of 24. Circle all of the numbers that are factors and draw a square around numbers that are multiples. Explain what they are factors and multiples of, and how you know.
- My friend says that if you know the answer to 6 × 17, you must also know the answer to 17 × 6. Is he correct? Why/why not?
- My friend says that if you know that 6 × 17 = 102, you must also know the answer to 102 ÷ 6? Is he correct? Why/why not?

These questions are chosen because they explore key aspects of multiplicative thinking, the understanding (or otherwise) of which is likely to provide an indication of a student's level of thinking.

Overview of Results

The responses from the interviews and questionnaires made for some interesting overall observations. First, responses from the Year Five cohort at School A and the Year Six cohort at School B revealed a wide range of conceptual understanding. Second, responses to the questionnaire administered at School C revealed that the three class groups within each year level had varying levels of understanding. While there were large variations within each year level, a similar range was evident between year levels, and indeed, within each class group. This paper suggests that the differences may have resulted, at least to some extent, from different pedagogies, teaching styles, and/or may reflect different stages of development of children's understandings of multiplicative concepts. After all, Siemon et al. (2011) have noted that multiplicative thinking usually does not fully develop until the early secondary years.

Specific Results and Discussion

School A and School B

The purpose of this paper is not to compare performance of different school cohorts against one another or different sections of school cohorts against one another. Rather it seeks to identify aspects of multiplicative thinking that might be evident or otherwise in different children and to understand why that might be so. Hence interview results from Schools A and B are combined into one set. Table 1 presents a summary of responses to the five illustrative questions listed above for the Year Five cohort from School A and the Year Six cohort from School B (n = 38).

Table 1 - *Summary of Responses from Schools A and B*

Mathematical understanding demonstrated by responses to listed questions	School A Year Five School B Year Six (n=38)
Identifies numbers in multiplication fact as 'group size' and 'number of groups'.	39%
Represents given multiplication fact as an array.	34%
Defines 'factor' and 'multiple' and/or identifies factors and multiples in given number fact.	63%
Explains commutative property in a conceptual way and/or demonstrates it using an array.	29%
Explains inverse relationship in a conceptual way based on number of groups and group size	50%

It is also worth noting that of the 38 children in the combined sample, ten (26%) responded correctly to four or five of the above questions and a further seven (18%) responded correctly to three of the questions. This seems to indicate that approximately one quarter of the sample demonstrated a strong level of conceptual understanding of the selected aspects of multiplicative thinking and a smaller proportion showed a reasonable level of understanding. However, over half the children in the sample could only respond appropriately to two or less of the selected questions. This suggests that there is a wide range of understanding across the sample.

Some Typical Strong Responses

Typical responses demonstrating a strong level of conceptual understanding of the commutative property of multiplication include the following:

- Student Dylan – "It doesn't really matter which way it is – seventeen groups of six is the same as six groups of seventeen". He then used tiles to make three groups of five and five groups of three, and also rearranged twelve tiles saying "I just put them into a three by four grid – it's the same as a four by three".
- Student Dean – "It's just the same . . . you just flip it around". He then used tiles to make a three by five array and rotated the array to explain his point.
- Similarly, the following exchange during the interview with Student Jason shows some connection of ideas around the inverse relationship between multiplication and division, sharing into equal groups, and arrays.
- When discussing the division fact 24 ÷ 3, Jason showed it as an array and said, "Then I'm going to split it up into threes, because I'm going to see how many groups of three

> I can have in 24". Also said, when asked what the answer would be, "I started with knowing that how many threes go into 15 and that's five, then I counted by threes to get 18, 21, 24". He also said, "If I had 3 times 4 it would be 12. If I had 12 divided by 3 it would be 4". He also gave a similar example with "8 groups of 3 = 24, So 24÷8 = 3"

Such connected discussion seems to demonstrate a sound understanding of the concepts involved.

Responses Indicating Partial Understanding

The apparent lack of conceptual understanding in the responses of some children is of interest. It is difficult to draw conclusions about the depth of some children's conceptual understanding given the absence of links and connections between responses to different questions. That is, some children show some understanding of a particular idea which would lead one to reasonably expect they would show an understanding of related concepts. However, this was often not the case.

It is well accepted that the array is a powerful representation of the multiplicative situation (Jacob & Mulligan, 2014). However, while two of the children (Ellie and Tilly) drew an array to represent the given number fact, neither of them could explain why the commutative property works, in terms of the array. Rather, they said that the numbers were 'swapped around' (Tilly) or 'you've just swapped them around' (Ellie). Also, of the thirteen children who drew an array, only seven of them could describe factors and multiples.

Similarly, while 63% of the children (n = 24) could adequately describe factors and multiples and their roles in the multiplicative situation, only five of them talked about the 7×6 number fact in terms of group size and number of groups. Further to that, 39% of the children (n = 15) described the number fact in terms of group size and number of groups, yet only five of that group also drew an array. Some of the children (n = 11) adequately explained the commutative property and half (n = 19) explained the inverse relationship between multiplication and division. However, not all of the eleven children who explained the commutative property could also explain the inverse relationship. This is interesting because the ideas underpinning those interview tasks are inextricably linked – that is, group size/number, the factor-factor-multiple relationship, the representation as an array, the commutative property, and the inverse relationship. Hence, it might be reasonably expected that there would be more children who could perform well on all or most of the items, or on none (or very few) of them.

School A and B Implications

The inferences that can be drawn from the School A and B data suggest that the connections between those important ideas need to be made clear and more explicit so that a mutual understanding of them can be developed. This is supported by the observations that: some students drew an array; some others could explain factors and multiples; and some others could explain a multiplication fact in terms of group size and number. Perhaps it is because there has been passing mention made of these key ideas, rather than sustained and explicit teaching of them. For example, the fact that less than one third of the children could explain the commutative property in a conceptual way gives rise to questions about how the commutative property may have been taught. Perhaps it is also attributable to the fact that children's understanding of the multiplicative situation is developing and in a state of flux. After all, it has been noted (Siemon et al., 2011) that multiplicative thinking is a concept that does not fully develop until the secondary years around the age of fourteen and the students involved here are several years younger than that.

The lack of sustained teaching may also be because the teachers simply do not appreciate the critical importance of the ideas of factor/multiple, group size/number, and the use of the array. Hence they may have taught some of the ideas once, assuming that such exposure is adequate when that is clearly not the case.

School C

In School C, the questionnaire was administered to 180 children in Years Four, Five and Six. Table 2 represents the responses from the three year levels in School C to questions about the same concepts as shown in Table 1.

Table 2 - *Comparison of Responses from Different Year Levels at School C*

Mathematical understanding demonstrated by responses to listed questions	Year 4	Year 5	Year 6
Identifies numbers in multiplication fact as 'group size' and 'number of groups'.	14%	9%	23%
Represents given multiplication fact as an array.	28%	32%	39%
Defines 'factor' and 'multiple' and/or identifies factors and multiples in given number fact.	3%	15%	14%
Explains commutative property in a conceptual way and/or demonstrates it using an array.	0%	2%	5%
Explains inverse relationship in a conceptual way based on number of groups and group size	10%	1%	12%

In general it would probably be expected that the Year Six children would perform better than the Year Five children who would in turn perform better than the Year Four children. However, as can be seen, this is not always the case and even where it is, one would perhaps expect the comparative performance of the older children to be markedly better than it is.

Of more interest is the comparison within each year level in School C, as shown in Table 3 which shows responses from children in the three Year Four classes. Here, there are marked differences in the responses from different class groups, particularly in relation to the first two questions. It is indeed surprising that no child in Class 4A could identify 'group size' and 'number of groups' in multiplication facts when over a third (35%) of Class 4C could do so. Even more intriguing is that nearly two thirds (62%) of Class 4A drew an array to show a multiplication fact when not one child in Class 4C did that. As well, very few children in Class 4B responded correctly on any of the five questions. What does this indicate?

Table 3 - Comparison within Year Levels in School C

Mathematical understanding demonstrated by responses to listed questions	4A	4B	4C
Identifies numbers in multiplication fact as 'group size' and 'number of groups'.	0%	6%	35%
Represents given multiplication fact as an array.	62%	17%	0%
Defines 'factor' and 'multiple' and/or identifies factors and multiples in given number fact.	0%	6%	5%
Explains commutative property in a conceptual way and/or demonstrates it using an array.	0%	0%	0%
Explains inverse relationship in a conceptual way based on number of groups and group size	14%	0%	15%

School C Implications

Classes at School C are not streamed on ability. Hence, it seems reasonable to assume that the variation in responses may be due to different teaching occurring among the three Year Four classes. Perhaps there has been a clear emphasis in Class 4A on the use of arrays, rather than showing multiplication facts as separate groups. It is also worth noting that the responses from Class 4A (62%) to the array question were the highest of any class in the school – only one Year Six class (53%) and one Year Five class (50%) recorded a similar level of correct responses. Similarly, the teaching in Class 4C is likely to have emphasized the notion of 'group size' and 'number of groups' in the multiplicative situation. Again, Class 4C's response (35%) is the highest recorded of all classes with only one Year Six class (33%) recording a similar level of correct answers. However, it seems reasonable to imply that there is a need for explicit teaching of the connections between the five related ideas in the multiplicative situation, something which seems to be reflected in the responses from Schools A and B as well.

General Implications

The five selected questions from the interview and questionnaire represent less than a quarter of the full instrument yet the data generated from just three sets of children have provided plenty of food for thought. There are two main observations that can be made from the presented data. First, there are considerable differences in the levels of understanding of multiplicative concepts shown by two groups (Schools A and B) that were interviewed. Some children displayed more connected understanding than did others. Second, there is considerable difference in responses among classes in the same year level at the same school (School C) where the questionnaire was administered. In seeking reasons for this, it is reasonable to infer that the differences may be due to pedagogies.

The differences in responses are quite stark at times and the relative connectedness in the thinking of some children in the combined cohort from Schools A and B suggests that connections between ideas may have been made more explicit in some classes compared to others. At least, it is likely that some children from the School A/B group have been encouraged to justify, explain, and reason about their ideas, as well as interpret those of others. Apart from the differences in responses, some children were more forthcoming and articulate which suggests that they may have been more accustomed to discussing mathematical ideas than other children from the same combined group who were often unable to elaborate their answers. Also, in responding to questions other than those reported here, some children were reluctant and/or unable to depart from quite procedural responses which was not the case with other children.

Conclusions

In conclusion, there are implications for teaching in terms of what can be done to help children develop key multiplicative concepts in a connected way. The evidence presented here suggests that such pedagogical practices exist but may not be sufficiently widespread. Such teaching could include the following:

- Explicitly teach that the multiplicative situation is based on the number of equal groups and the size of each group.
- Develop an understanding of the terms factor and multiple through the use of arrays, and explicitly use them as 'mathematical language'.
- Teach multiplication and division simultaneously, not separately.
- Develop the commutative property through the use of arrays and physically show the 'x' rows of 'y' gives the same result as 'y' rows of 'x'.

If teachers view multiplication and division as different ways of representing 'the multiplicative situation', rather than as separate entities, the links and connections between the ideas discussed in this paper can be made explicit for children. When those connections are clearly understood, ideas such as the inverse relationship and the commutative property become much easier to grasp.

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