2017

On the combinatorics of Demoulin transforms and (discrete) projective minimal surfaces

A McCarthy
The University of Notre Dame Australia, alan.mccarthy@nd.edu.au

W Schief

This article was originally published as:
McCarthy, A., & Schief, W. (2017). On the combinatorics of Demoulin transforms and (discrete) projective minimal surfaces. *Discrete and Computational Geometry, 57* (1), 215-230.

Original article available here:
https://doi.org/10.1007/s00454-016-9827-x

Noname manuscript No.
(will be inserted by the editor)

On the combinatorics of the classical Demoulin transformation and (discrete) projective minimal surfaces.

A. McCarthy · W.K. Schief

Received: date / Accepted: date

Abstract The classical Demoulin transformation is examined in the context of discrete differential geometry. We show that iterative application of the Demoulin transformation to a seed projective minimal surface generates a $\mathbb{Z}^2$ lattice of projective minimal surfaces. Known and novel geometric properties of these Demoulin lattices are discussed and used to motivate the notion of lattice Lie quadrics and associated discrete envelopes and the definition of the class of discrete projective minimal and Q-surfaces (PMQ-surfaces). We demonstrate that the even and odd Demoulin sublattices encode a two-parameter family of pairs of discrete PMQ-surfaces with the property that one discrete PMQ-surface constitute an envelope of the lattice Lie quadrics associated with the other.

Keywords Demoulin transformation · Projective minimal surfaces · Discrete differential geometry

Mathematics Subject Classification (2000) 53A20 · 37K25 · 37K35

A. McCarthy
School of Mathematics and Statistics, The University of New South Wales, Sydney, NSW 2052, Australia
Tel.: +612-9385-7076
Fax: +612-9385-7123
E-mail: a.mccarthy@unsw.edu.au

W.K. Schief
School of Mathematics and Statistics, The University of New South Wales, Sydney, NSW 2052, Australia

Australian Research Council Centre of Excellence for Mathematics and Statistics of Complex Systems, School of Mathematics and Statistics, The University of New South Wales, Sydney, NSW 2052, Australia
Tel.: +612-9385-3131
E-mail: w.schief@unsw.edu.au

1 Introduction

The intimate connection between classical differential geometry and its discrete counterpart (discrete differential geometry) and the theory of continuous and discrete integrable systems has been well documented (see, e.g., [2, 9] and references therein). Moreover, Bäcklund transformations provide a link between classical and discrete differential geometry and, at the algebraic level, the underlying partial differential and difference equations such as the (discrete) Gauss-Weingarten equations in the case of (discrete) surface theory. For instance, iterative application of the classical Bäcklund transformation to surfaces of constant negative Gaussian curvature and the sine-Gordon equation as the Gauss equation not only generates discrete analogues of these surfaces [11,14,1] but also gives rise to Hirota's discrete sine-Gordon equation [5].

In general, the application of a Bäcklund transformation to an integrable class of surfaces, that is, a class of surfaces governed by an integrable system, requires the solution of a linear system of differential equations (Lax pair) which depends on a Bäcklund (spectral) parameter [9]. In particular, a Bäcklund transform of a seed surface consists of a family of surfaces which is labelled by the Bäcklund parameter. This applies, for instance, to the class of minimal surfaces in projective differential geometry [9,3]. However, given any projective minimal surface, there exists an alternative procedure which may be used to generate an infinite number of projective minimal surfaces. This classical transformation, which bears the name of Demoulin [10], is purely algebraic in nature and does not involve a Lax pair or a Bäcklund parameter. It is the aim of this paper to record known and important novel properties of the classical Demoulin transformation with a view to highlighting its significance in discrete differential geometry.

A surface Σ in a three-dimensional projective space $\mathbb{RP}^3$ comes with a two-parameter family of Lie quadrics [3,4], each of which has second-order contact with the surface at the corresponding point. By definition of the Lie quadric, the surface Σ is an envelope of the family of Lie quadrics but, generically, there exist four additional envelopes known as Demoulin transforms of Σ. Remarkably, the Demoulin transforms of projective minimal surfaces are projective minimal [10,8] so that iterative application of the Demoulin transformation generates an infinite number of projective minimal surfaces. Even though, in principle, there exist sixteen second generation Demoulin transforms of a projective minimal surface Σ, it is known [10,12] that, generically, due to coincidence, there exist only nine distinct second generation Demoulin transforms and one of them is the seed surface Σ. Hence, it is natural to determine the cardinality of the set of projective minimal surfaces generated by iterative application of the Demoulin transformation and how the individual surfaces in this set are combinatorially related.

It turns out that the Demoulin transformation generically generates a set of projective minimal surfaces of $\mathbb{Z}^2$ combinatorics which we term a Demoulin lattice. Any point on the seed surface labelled by (x, y) is mapped to corresponding points on its transforms and, hence, the Demoulin transformation generates a two-parameter family of $\mathbb{Z}^2$ lattices in $\mathbb{RP}^3$. Furthermore, due to the Weingarten relation [10] between a projective minimal surface and (some of) its second generation transforms, the even and odd sublattices of any $\mathbb{Z}^2$ lattice in this family have planar stars. Hence, for any fixed (x, y), the Demoulin lattice decomposes into two discrete asymptotic nets. The latter have been used extensively in discrete differential geometry as a natural discretisation of asymptotic lines on hyperbolic surfaces [2].

Remarkably, the Lie quadrics attached to the projective minimal surfaces of the Demoulin lattice may also be interpreted as "lattice Lie quadrics" associated with the discrete asymptotic nets encoded in the even and odd sublattices. Hence, we introduce the notion of discrete envelopes of lattice Lie quadrics and show that any discrete asymptotic net associated with the even Demoulin sublattice may be regarded as an envelope of the lattice Lie quadrics of the corresponding discrete asymptotic net associated with the odd Demoulin sublattice and vice versa. Here, we exploit the theory of hyperbolic nets developed in detail in [6,7].

In view of the classical theory, the above analysis naturally leads to the definition of discrete PMQ-surfaces which are discretisations of either projective minimal surfaces or so-called Q-surfaces [3]. This is motivated by an important theorem in projective differential geometry [3,10] which states that the asymptotic lines on a surface and at least one Demoulin transform correspond if and only if the surface is either projective minimal or of Q type. We then prove the key theorem which asserts that the discrete asymptotic nets encoded in the Demoulin lattice constitute discrete PMQ-surfaces.

2 Demoulin transformations

2.1 Algebraic classification of projective minimal surfaces

Consider a surface Σ in a three-dimensional projective space $\mathbb{RP}^3$ represented by $[\boldsymbol{r}] : \mathbb{R}^2 \to \mathbb{RP}^3$ in terms of asymptotic coordinates (x, y) so that the vector of homogeneous coordinates $\boldsymbol{r} \in \mathbb{R}^4$ satisfies a pair of linear equations

$$\boldsymbol{r}_{xx} = p\boldsymbol{r}_y + \pi\boldsymbol{r} + \sigma\boldsymbol{r}_x, \quad \boldsymbol{r}_{yy} = q\boldsymbol{r}_x + \xi\boldsymbol{r} + \chi\boldsymbol{r}_y.$$

Then, it is well known [9,3,10] that particular homogeneous coordinates, known as the Wilczynski lift, may be chosen such that the functions σ and χ vanish. Hence, the remaining coefficients of the "projective Gauss-Weingarten equations" may be parametrised according to

$$\boldsymbol{r}_{xx} = p\boldsymbol{r}_y + \frac{1}{2}(V - p_y)\boldsymbol{r}, \quad \boldsymbol{r}_{yy} = q\boldsymbol{r}_x + \frac{1}{2}(W - q_x)\boldsymbol{r},$$

in terms of functions p, q, V and W. The latter are constrained by the compatibility condition $\boldsymbol{r}_{xxyy} = \boldsymbol{r}_{yyxx}$ which leads to the "projective Gauss-Mainardi-

Codazzi" equations

$$p_{yyy} - 2p_y W - pW_y = q_{xxx} - 2q_x V - qV_x \tag{1}$$
$$W_x = 2qp_y + pq_y \tag{2}$$
$$V_y = 2pq_x + qp_x. \tag{3}$$

We note that the Wilczynski lift is unique up to the group of transformations

$$x \to f(x), \quad y \to g(y), \quad \boldsymbol{r} \to \sqrt{f'(x)g'(y)}\,\boldsymbol{r}$$

with

$$p \to p\frac{g'(y)}{[f'(x)]^2}, \qquad q \to q\frac{f'(x)}{[g'(y)]^2} \tag{4}$$

$$V \to \frac{V + S(f)}{[f'(x)]^2}, \quad W \to \frac{W + S(g)}{[g'(y)]^2}, \tag{5}$$

where S denotes the Schwarzian derivative

$$S(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

The quadratic form

$$pq\,dxdy$$

is a projective invariant and is known as the projective metric. Throughout the paper, we shall assume that Σ is not ruled, i.e., $pq \neq 0$. In view of the structure of relation (1), we define functions α and β by

$$\alpha = p^2 W - pp_{yy} + \frac{p_y^2}{2}, \quad \beta = q^2 V - qq_{xx} - \frac{q_x^2}{2}$$

so that the Gauss-Mainardi-Codazzi equations (1)-(3) adopt the form

$$\frac{\alpha_y}{p} = \frac{\beta_x}{q}$$
$$(\ln p)_{xy} = pq + \frac{A}{p}, \quad A_y = -p\left(\frac{\alpha}{p^2}\right)_x$$
$$(\ln q)_{xy} = pq + \frac{B}{q}, \quad B_x = -q\left(\frac{\beta}{q^2}\right)_y.$$

This is directly verified by eliminating the functions A and B. The above parametrisation turns out to be convenient in connection with the classification of projective minimal surfaces.

Definition 1 A surface Σ in $\mathbb{RP}^3$ is said to be projective minimal if it is critical for the area functional $\iint pq\,dxdy$.

Theorem 1 ([13]) *A surface Σ in $\mathbb{RP}^3$ is projective minimal if and only if*

$$\frac{\alpha_y}{p} = \frac{\beta_x}{q} = 0.$$

A projective minimal surface is said to be

(a) generic if $\alpha \neq 0$ and $\beta \neq 0$.
(b) of Godeaux-Rozet type if $\alpha \neq 0$, $\beta = 0$ or $\alpha = 0$, $\beta \neq 0$.
(c) of Demoulin type if $\alpha = \beta = 0$. If, in addition, $p = q$, then Σ is said to be of Tzitzéica type.

We note that, using a gauge transformation of the form (4), (5), we may normalise α and β to be one of -1, 1 or 0.

2.2 Geometric classification of projective minimal surfaces

The above algebraic classification admits a corresponding geometric interpretation which involves Lie quadrics and their envelopes.

Definition 2 Let $[\boldsymbol{r}] : \mathbb{R}^2 \to \mathbb{RP}^3$ be a parametrisation of a surface Σ in terms of asymptotic coordinates. Let $\boldsymbol{p} = \boldsymbol{r}(x, y)$ be a point on Σ and let $\boldsymbol{p}_\pm$ be two additional points on the x-asymptotic line passing through $\boldsymbol{p}$, given by $\boldsymbol{p}_\pm = \boldsymbol{r}(x \pm \epsilon, y)$. Let $l_\pm$ and l be the three lines tangent to the y-asymptotic lines at $\boldsymbol{p}_\pm$ and $\boldsymbol{p}$ respectively. These uniquely define a quadric Q_ϵ containing them as rectilinear generators. The Lie quadric at (x, y) is then the unique quadric defined by

$$Q(x, y) = \lim_{\epsilon \to 0} Q_\epsilon(x, y).$$

Remark 1 *It turns out that the above definition of a Lie quadric is symmetric in x and y, that is, interchanging x-asymptotic lines and y-asymptotic lines leads to the same Lie quadric Q.*

An explicit representation of the Lie quadric Q (at a point) is given below [3,4]. For brevity, in the following, we do not distinguish between a Lie quadric in $\mathbb{RP}^3$ and its representation in the space of homogeneous coordinates $\mathbb{R}^4$.

Theorem 2 *The Lie quadric Q admits the parametrisation*

$$Q = \mathfrak{n} + \mu\mathfrak{r}^1 + \nu\mathfrak{r}^2 + \mu\nu\mathfrak{r},$$

where μ and ν parametrise the two families of generators of Q and $\{\mathfrak{r}, \mathfrak{r}^1, \mathfrak{r}^2, \mathfrak{n}\}$ is the Wilczynski frame given by

$$\mathfrak{r} = \boldsymbol{r}, \quad \mathfrak{r}^1 = \boldsymbol{r}_x - \frac{q_x}{2q}\boldsymbol{r}, \quad \mathfrak{r}^2 = \boldsymbol{r}_y - \frac{p_y}{2p}\boldsymbol{r}$$
$$\mathfrak{n} = \boldsymbol{r}_{xy} - \frac{p_y}{2p}\boldsymbol{r}_x - \frac{q_x}{2q}\boldsymbol{r}_y + \left(\frac{p_y q_x}{4pq} - \frac{pq}{2}\right)\boldsymbol{r}.$$

We note that the lines $(\mathfrak{r},\,\mathfrak{r}^1)$ and $(\mathfrak{r},\,\mathfrak{r}^2)$ are tangent to Σ. The line $(\mathfrak{r},\,\mathfrak{n})$ is transversal to Σ and plays the role of a projective normal. It is known as the first directrix of Wilczynski.

Definition 3 A surface Ω parametrised by $[\omega] : \mathbb{R}^2 \to \mathbb{RP}^3$ is an envelope of the two parameter family of Lie quadrics $\{Q(x,y)\}$ associated with a surface Σ if $\omega(x,y) \in Q(x,y)$ such that Ω touches $Q(x,y)$ at $\omega(x,y)$.

We note that, in particular, Σ is itself an envelope of $\{Q\}$. Generically, there exist four additional envelopes as stated below [3].

Theorem 3 *If $\alpha,\ \beta \geq 0$ then the Lie quadrics $\{Q\}$ possess four real additional envelopes*

$$\begin{aligned}
\omega_{++} &= \mathfrak{n} + \hat{\mu}\mathfrak{r}^1 + \hat{\nu}\mathfrak{r}^2 + \hat{\mu}\hat{\nu}\mathfrak{r} \\
\omega_{+-} &= \mathfrak{n} + \hat{\mu}\mathfrak{r}^1 - \hat{\nu}\mathfrak{r}^2 - \hat{\mu}\hat{\nu}\mathfrak{r} \\
\omega_{-+} &= \mathfrak{n} - \hat{\mu}\mathfrak{r}^1 + \hat{\nu}\mathfrak{r}^2 - \hat{\mu}\hat{\nu}\mathfrak{r} \\
\omega_{--} &= \mathfrak{n} - \hat{\mu}\mathfrak{r}^1 - \hat{\nu}\mathfrak{r}^2 + \hat{\mu}\hat{\nu}\mathfrak{r},
\end{aligned}$$

where

$$\hat{\mu} = \sqrt{\frac{\alpha}{2p^2}}, \quad \hat{\nu} = \sqrt{\frac{\beta}{2q^2}}.$$

These are distinct if $\alpha, \beta \neq 0$.

Remark 2 *The above envelopes are called the Demoulin transforms of Σ. We denote them by Σ_{++}, Σ_{+-}, Σ_{-+} and Σ_{--}.*

As indicated in the above theorem, the expressions for $\hat{\mu}$ and $\hat{\nu}$ imply that whether α and β vanish or not is related to the distinct number of envelopes. The geometric interpretation of the algebraic classification (a)-(c) is then that a projective minimal surface Σ is

(a) generic if the set of Lie quadrics $\{Q\}$ has four distinct additional envelopes.
(b) of Godeaux-Rozet type if $\{Q\}$ has exactly two distinct additional envelopes.
(c) of Demoulin type if $\{Q\}$ has exactly one additional envelope.

Remark 3 *Theorem 3 implies that a surface Σ in $\mathbb{RP}^3$ is necessarily projective minimal if there exist less than four additional distinct envelopes. Specifically, if the Lie quadrics of Σ have only two additional distinct envelopes then Σ is of Godeaux-Rozet type. If the Lie quadrics of Σ have only one additional envelope then Σ is of Demoulin type.*

Remark 4 *For any fixed (x,y), the points $\omega_{++}(x,y)$, $\omega_{+-}(x,y), \omega_{--}(x,y)$ and $\omega_{-+}(x,y)$ of the Demoulin transforms of Σ may be regarded as the vertices of a quadrilateral*

$$[\omega_{++}(x,y),\, \omega_{+-}(x,y),\ \omega_{--}(x,y), \omega_{-+}(x,y)]$$

which is known as the Demoulin quadrilateral. Then, the parametrisation of the envelopes in Theorem 3 shows that the extended edges $[\omega_{++}(x,y),\, \omega_{+-}(x,y)]$, $[\omega_{+-}(x,y),\, \omega_{--}(x,y)]$, $[\omega_{--}(x,y),\, \omega_{-+}(x,y)]$ *and* $[\omega_{-+}(x,y),\, \omega_{++}(x,y)]$ *are generators of the Lie quadric* $Q(x,y)$*. This induces a natural pairing between the Demoulin transforms corresponding to the diagonal of the Demoulin quadrilateral. We thus call* Σ_{++} *and* Σ_{--} *opposite transforms and, similarly,* Σ_{+-} *and* Σ_{-+} *are opposite transforms.*

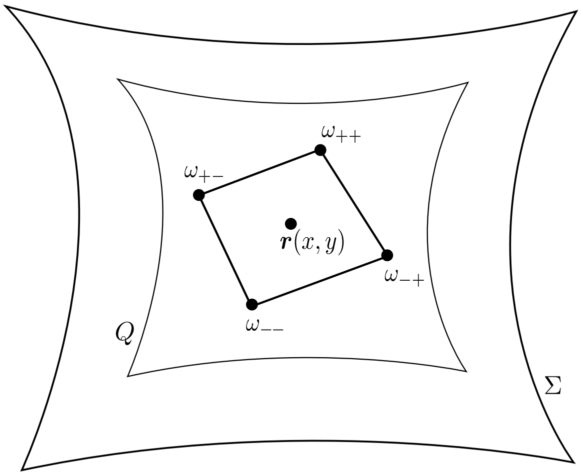

Fig. 1: The Demoulin quadrilateral

It turns out that the Demoulin transformation acts within the class of projective minimal surfaces and within the classes (a)-(c) [10,8].

Theorem 4 *Let* Σ *be a projective minimal surface. Then, each of its Demoulin transforms is projective minimal. Moreover, the number* $n \in \{1,2,4\}$ *of distinct Demoulin transforms of* Σ *is preserved by the Demoulin transformation. In particular, if* Σ *is of Godeaux-Rozet type then each of its Demoulin transforms is of Godeaux-Rozet type. If* Σ *is of Demoulin type then its transform is of Demoulin type.*

2.3 Iteration of the Demoulin transformation

Let Σ be a generic projective minimal surface. Then, it has four distinct Demoulin transforms which we call first generation Demoulin transforms. Each of these transforms again has four Demoulin transforms which we call second generation transforms. We arrange Σ and the four first generation transforms in a star of a $\mathbb{Z}^2$ lattice with coordinates $(n_1,\, n_2)$ in such a way that opposite transforms are placed at vertices which correspond to an increment or decrement of the same coordinate, and Σ is placed at the centre of the star. We denote an increment of n_k by a subscript k and a decrement by a subscript $\bar{k}$ as in Figure 2.

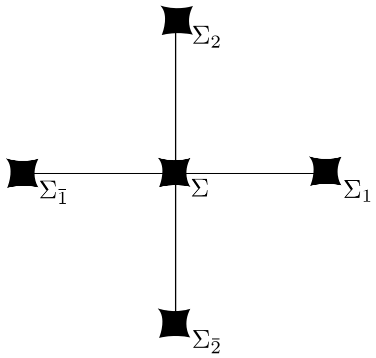

Fig. 2: First generation Demoulin transforms

The following theorem expresses the remarkable known fact that only nine of the sixteen second generation transforms of a generic projective minimal surface Σ are distinct, one of which is Σ [10,12].

Theorem 5 *Let Σ be a generic projective minimal surface and Σ_θ its Demoulin transforms with $\theta = 1,\ \bar{1},\ 2,\ \bar{2}$. Then,*

(i) Σ is a Demoulin transform of each Σ_θ,
(ii) Σ_θ and Σ_λ, $\theta \neq \lambda$, have a common transform different from Σ if and only if they are not opposite transforms.

The above theorem implies that Σ_1 and Σ_2 have a common transform which we denote by Σ_{12}. Similarly, we have $\Sigma_{\bar{1}2}$, $\Sigma_{\bar{1}\bar{2}}$ and $\Sigma_{1\bar{2}}$ as displayed in Figure 3. The remaining Demoulin transform of Σ_1 is denoted by Σ_{11} and, similarly, we define $\Sigma_{\bar{1}\bar{1}}$, Σ_{22} and $\Sigma_{\bar{2}\bar{2}}$. These are then the eight distinct second generation transforms different from Σ as stated above.

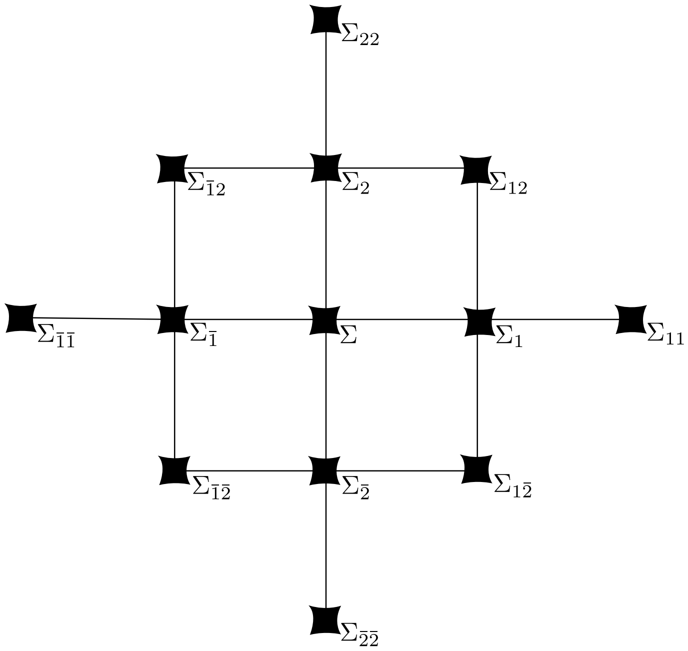

Fig. 3: The combinatorics of first and second generation Demoulin transforms

Even though Figure 3 is merely a combinatorial re-arrangement of the known relations between a generic projective minimal surface and its first and second generation Demoulin transforms, it gives rise to the novel observation that iterative application of the Demoulin transformation leads to an infinite number of projective minimal surfaces which may be combinatorially attached to the vertices of a $\mathbb{Z}^2$ lattice.

Theorem 6 *The set of all (iterated) Demoulin transforms of a generic projective minimal surface Σ forms a $\mathbb{Z}^2$ lattice of projective minimal surfaces.*

Proof *Consider the second generation transforms Σ_{12} and Σ_{22}. Then, by Theorem 5, there exists a common Demoulin transform Σ_γ of Σ_{12} and Σ_{22}. Moreover, Theorem 5 also implies that this surface is distinct from Σ_2 and thus distinct from its opposite transform with respect to Σ_{12}. Thus, there exist only two possibilities for Σ_γ, namely Σ_1 and its opposite transform. If Σ_γ were Σ_1 then Σ_{22} would be a common transform of Σ_1 and Σ_2 and, hence, Σ_{12} would coincide with Σ_{22} which would be a contradiction since these are distinct surfaces. Hence, Σ_γ is the opposite transform which we denote by Σ_{122}. Similarly, the common Demoulin transform between $\Sigma_{\bar{1}2}$ and Σ_{22} can be seen to be the opposite transform to Σ_{122}. We denote the remaining transform of Σ_{22} by Σ_{222}. Continuing in this manner, we can generate a $\mathbb{Z}^2$ lattice of Demoulin transforms of Σ, part of which is shown in Figure 4.*

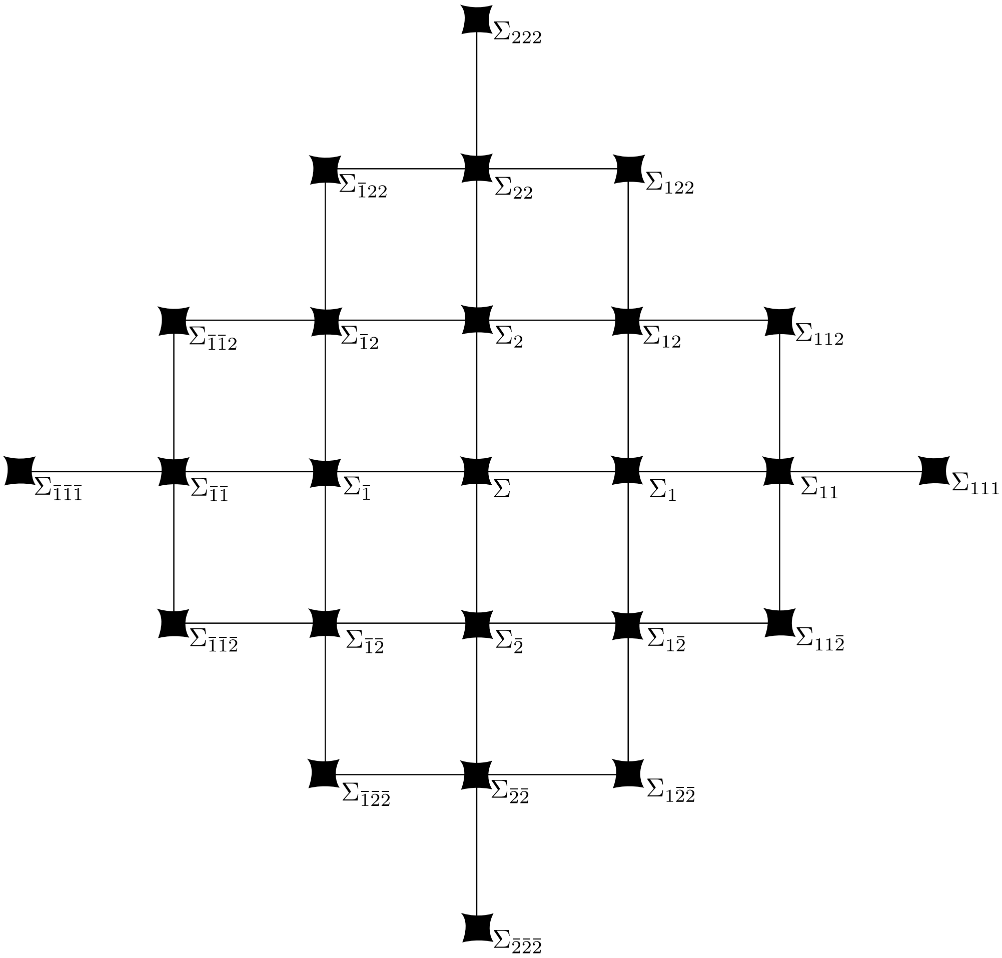

Fig. 4: Part of the $\mathbb{Z}^2$ Demoulin lattice

We will now examine in detail the combinatorial and geometric implications of Theorem 6.

3 The Demoulin lattice

We denote the set of projective minimal surfaces in $\mathbb{RP}^3$ by $\mathcal{M}$ and the corresponding lattice in Theorem 6 by

$$\Sigma : \mathbb{Z}^2 \to \mathcal{M},$$

where $\Sigma(0,0)$ is the seed surface and $\Sigma(n_1, n_2)$ is a Demoulin transform of generation $|n_1|+|n_2|$. If $\Sigma(0,0)$ is represented by $[\boldsymbol{r}] : \mathbb{R}^2 \to \mathbb{RP}^3$ then iterative application of the Demoulin transformation generates a map of the form

$$\boldsymbol{r} : \mathbb{R}^2 \times \mathbb{Z}^2 \to \mathbb{R}^4, \tag{6}$$

where, for a fixed $(n_1, n_2) \in \mathbb{Z}^2$, $\boldsymbol{r}$ describes a projective minimal surface and, for a fixed (x, y), $\boldsymbol{r}$ describes a $\mathbb{Z}^2$ lattice of points on projective minimal surfaces related by Demoulin transformations. We will sometimes suppress x and y or n_1 and n_2 when the context is clear in order to think of $\boldsymbol{r}$ as a map $\boldsymbol{r} : \mathbb{Z}^2 \to \mathbb{R}^4$ or $\boldsymbol{r} : \mathbb{R}^2 \to \mathbb{R}^4$. For a fixed (x, y), each lattice point is the point of contact of a Lie quadric and the corresponding surface. Thus, we can define a two-parameter family of lattices of Lie quadrics

$$Q : \mathbb{R}^2 \times \mathbb{Z}^2 \to \{\text{quadrics in } \mathbb{RP}^3\}. \tag{7}$$

It turns out that, for a fixed (x, y), the two sublattices corresponding to even and odd generation transforms of Σ have distinctive geometric properties.

3.1 Asymptotic sublattices and associated Lie quadrics

Theorem 7 ([10]) *Let Σ be a generic projective minimal surface. Then, the line connecting a point on Σ and the corresponding point on any one of its coincidental second generation Demoulin transforms (i.e., Σ_{12}, $\Sigma_{\bar{1}2}$, $\Sigma_{1\bar{2}}$ and $\Sigma_{\bar{1}\bar{2}}$) is tangent to both surfaces.*

Remark 5 *We note that by varying the x and y parameters on Σ, for each coincidental second generation Demoulin transform $\Sigma_{\theta\lambda}$, we obtain a two-parameter family of lines connecting Σ and $\Sigma_{\theta\lambda}$. Hence, these families of lines form W-congruences [10].*

As a result of Theorem 7, for a fixed (x, y), it is then natural to distinguish between the even and odd sublattices of $\boldsymbol{r} : \mathbb{Z}^2 \to \mathbb{R}^4$. Moreover, due to the existence of the W-congruences, each of these sublattices contains planar stars (regarded as objects in $\mathbb{RP}^3$). $\mathbb{Z}^2$ lattices whose stars are planar are termed discrete asymptotic nets. These are known [2] to be canonical discrete versions of surfaces parametrised in terms of asymptotic coordinates. Accordingly, the following theorem holds.

Theorem 8 *The lattice of Demoulin transforms of a projective minimal surface Σ, evaluated at a point (x, y), decomposes into two discrete asymptotic nets corresponding to even and odd generation Demoulin transforms.*

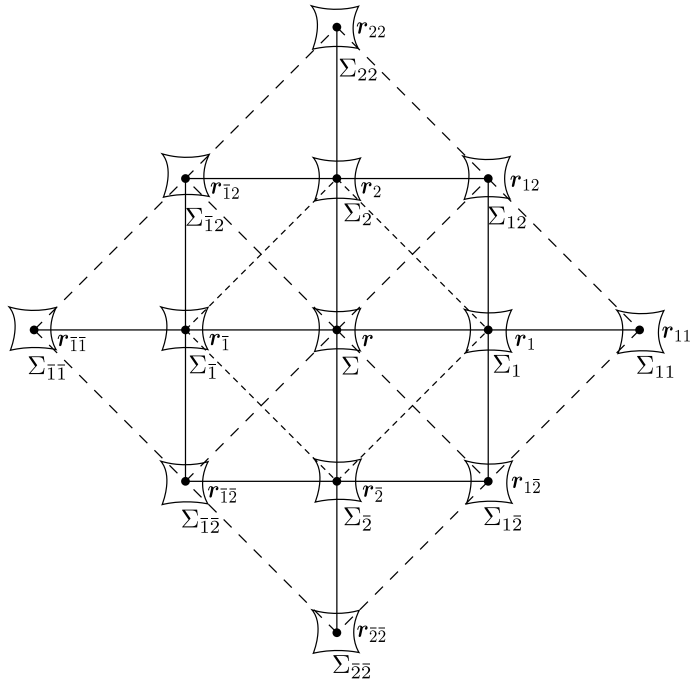

Fig. 5: A pair of discrete asymptotic nets (long and short dashes) obtained by restricting a Demoulin lattice of surfaces to a fixed point (x, y)

In particular, since the vertices $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\boldsymbol{r}_{\bar{1}}$ and $\boldsymbol{r}_{\bar{2}}$ of one sublattice are the vertices of the Demoulin quadrilateral associated with the Lie quadric Q combinatorially attached to the vertex $\boldsymbol{r}$ of the other sublattice, each sublattice is composed of Demoulin quadrilaterals. This raises the question as to the geometric nature of the relation between neighbouring quadrilaterals. A non-planar quadrilateral admits a one-parameter family of quadrics passing through the edges. Moreover, it is known [6] that, given a fixed member Q of this family, there exists exactly one quadric in a neighbouring family of quadrics whose tangent planes along the common edge coincide with those of Q. We call this relation between neighbouring quadrics the $\mathcal{C}^1$ property. Given two quadrics Q_b and Q_d, as in Figure 6, which satisfy the $\mathcal{C}^1$ condition with respect to a quadric Q_a, there exist two ways of generating the quadric Q_c arising from the $\mathcal{C}^1$ condition with respect to Q_b, and with respect to Q_d. Remarkably, the two quadrics generated in this way coincide [6]. Thus, for any fixed quadric Q in the one-parameter family of quadrics passing through a quadrilateral of a discrete asymptotic net, the $\mathcal{C}^1$ condition uniquely determines quadrics passing through all remaining quadrilaterals. This leads to the following definition.

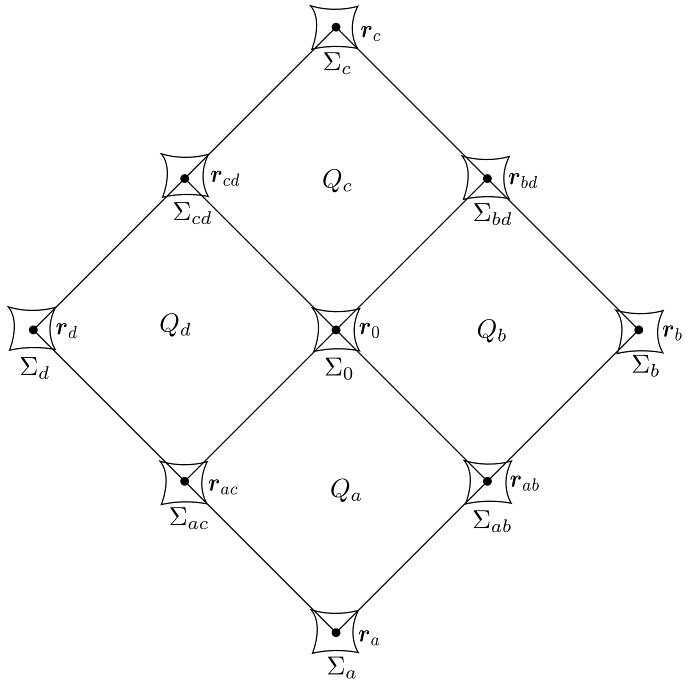

Fig. 6: Quadrics on an asymptotic sublattice

Definition 4 A set of quadrics $\{Q\}$ associated with the quadrilaterals of a discrete asymptotic net is termed a set of lattice Lie quadrics if the $\mathcal{C}^1$ condition holds for all neighbouring pairs of quadrics.

A justification for this terminology is as follows. Let $\boldsymbol{r}\colon \mathbb{Z}^2 \to \mathbb{R}^4$ be a discrete asymptotic net and $\{Q\}$ a set of associated lattice Lie quadrics. Let $\hat{Q}$ be the unique quadric passing through the lines $(\boldsymbol{r}\,\boldsymbol{r}_2)$, $(\boldsymbol{r}_1\,\boldsymbol{r}_{12})$ and $(\boldsymbol{r}_{11}\,\boldsymbol{r}_{112})$ (cf. Figure 7). We note that in the limit in the n_2 direction, the lines $(\boldsymbol{r}\,\boldsymbol{r}_2)$, $(\boldsymbol{r}_1\,\boldsymbol{r}_{12})$ and $(\boldsymbol{r}_{11}\,\boldsymbol{r}_{112})$ become tangent lines to the coordinate curves of a semi-discrete asymptotic net. Hence, by Definition 2, in the continuum limit, $\hat{Q}$ becomes a Lie quadric. Let $\boldsymbol{p}^0$ be a point on the line $(\boldsymbol{r}\,\boldsymbol{r}_2)$. Then, there exists a unique generator of $\hat{Q}$ through $\boldsymbol{p}^0$ which intersects the line $(\boldsymbol{r}_{11}\,\boldsymbol{r}_{112})$ at a point $\boldsymbol{p}^1$. $\boldsymbol{p}^1$ can be thought of as the intersection of the plane spanned by $\boldsymbol{p}^0$, $\boldsymbol{r}_1$ and $\boldsymbol{r}_{12}$ and the line $(\boldsymbol{r}_{11}\,\boldsymbol{r}_{112})$. On the other hand, let Q and Q_1 be neighbouring lattice Lie quadrics, as in Figure 7. Then, there exists a unique generator of Q through $\boldsymbol{p}^0$ which intersects the line $(\boldsymbol{r}_1\,\boldsymbol{r}_{12})$ at a point $\boldsymbol{p}^3$. Similarly, there exists a unique generator of Q_1 through $\boldsymbol{p}^3$ which intersects the line $(\boldsymbol{r}_{11}\,\boldsymbol{r}_{112})$ at a point $\boldsymbol{p}^2$. $\boldsymbol{p}^2$ can be thought of as the intersection of the tangent plane to Q at $\boldsymbol{p}^3$ and the line $(\boldsymbol{r}_{11}\,\boldsymbol{r}_{112})$. Then, as a result of the $\mathcal{C}^1$ condition, $\boldsymbol{p}^0$, $\boldsymbol{p}^2$, $\boldsymbol{r}_1$ and $\boldsymbol{r}_{12}$ are coplanar. Hence, $\boldsymbol{p}^1 = \boldsymbol{p}^2$. Therefore, it is evident that in the continuum limit the generators $(\boldsymbol{p}^0\,\boldsymbol{p}^1)$, $(\boldsymbol{p}^0\,\boldsymbol{p}^3)$ and $(\boldsymbol{p}^3\,\boldsymbol{p}^2)$ coincide. Hence, in the continuum limit, the quadrics Q, Q_1 and $\hat{Q}$ coincide.

Thus, the lattice Lie quadrics of the discrete asymptotic net formally converge to the Lie quadrics of the limiting continuous surface.

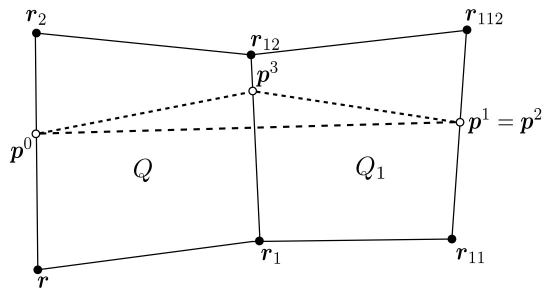

Fig. 7: A 3 × 2 patch of a discrete asymptotic net

Remarkably, the (surface) Lie quadrics attached to the vertices of one sublattice of the Demoulin lattice turn out to form a set of lattice Lie quadrics of the other sublattice. In order to show this, we need the following theorem.

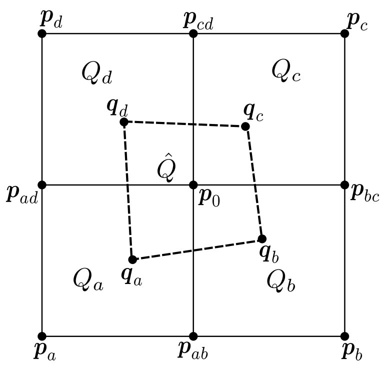

Fig. 8: A patch of a discrete asymptotic net with associated quadrics obeying the assumptions of Theorem 9

Theorem 9 *Let Σ be a 3×3 patch of a discrete asymptotic net containing four adjacent quadrilaterals $\Box_a$, $\Box_b$, $\Box_c$ and $\Box_d$ as in Figure 8. Let Q_a, Q_b, Q_c and Q_d be four quadrics in the one-parameter families of quadrics passing through $\Box_a$, $\Box_b$, $\Box_c$ and $\Box_d$ respectively. Given four points $\boldsymbol{q}_a$, $\boldsymbol{q}_b$, $\boldsymbol{q}_c$ and $\boldsymbol{q}_d$ on each of these quadrics (which do not coincide with the vertices of the corresponding quadrilaterals), if the edges of the quadrilateral $\widehat{\Box} = [\boldsymbol{q}_a, \boldsymbol{q}_b, \boldsymbol{q}_c, \boldsymbol{q}_d]$ touch the respective quadrics at the points $\boldsymbol{q}_a$, $\boldsymbol{q}_b$, $\boldsymbol{q}_c$ and $\boldsymbol{q}_d$, and there exists a quadric $\hat{Q}$ passing through $\widehat{\Box}$ which touches the planar star at the central vertex of Σ then the quadrics Q_a, Q_b, Q_c and Q_d satisfy the $\mathcal{C}^1$ condition.*

Proof *We label the vertices of Σ by $\boldsymbol{p}_a, \boldsymbol{p}_b, \boldsymbol{p}_c, \boldsymbol{p}_d, \boldsymbol{p}_{ab}, \boldsymbol{p}_{bc}, \boldsymbol{p}_{cd}, \boldsymbol{p}_{ad}$ and $\boldsymbol{p}_0$ as in Figure 8. We show that the $\mathcal{C}^1$ condition holds between Q_a and Q_b. The other $\mathcal{C}^1$ conditions are treated similarly. We may write $Q_a(\mu_\alpha, \nu_\alpha)$ as*

$$Q_a(\mu_\alpha, \nu_\alpha) = \boldsymbol{p}_a + \mu_\alpha \boldsymbol{p}_{ab} + \nu_\alpha \boldsymbol{p}_{ad} + \mu_\alpha \nu_\alpha \boldsymbol{p}_0.$$

Then, since $\boldsymbol{q}_a$ is a point on Q_a, there exist (μ_a, ν_a) and λ_a such that

$$Q_a(\mu_a, \nu_a) = \lambda_a \boldsymbol{q}_a,$$

hence

$$\boldsymbol{p}_a = \lambda_a \boldsymbol{q}_a - \mu_a \boldsymbol{p}_{ab} - \nu_a \boldsymbol{p}_{ad} - \mu_a \nu_a \boldsymbol{p}_0.$$

Furthermore, the assumption that $[\boldsymbol{q}_a, \boldsymbol{q}_b, \boldsymbol{q}_c, \boldsymbol{q}_d]$ touches Q_a, which means that the plane spanned by the edges $(\boldsymbol{q}_a\, \boldsymbol{q}_b)$ and $(\boldsymbol{q}_a\, \boldsymbol{q}_d)$ and the tangent plane of Q_a at $\boldsymbol{q}_a$ coincide, determines (μ_a, ν_a) in terms of $\boldsymbol{p}_{ab}$ and $\boldsymbol{p}_{ad}$. We can derive similar expressions for $\boldsymbol{p}_b$, $\boldsymbol{p}_c$ and $\boldsymbol{p}_d$ in terms of λ_b, λ_c and λ_d respectively. Here, and in the following, we suppress the details of the somewhat lengthy calculations and merely focus on the idea of the proof. Coplanarity of $\boldsymbol{p}_a$, $\boldsymbol{p}_{ab}$, $\boldsymbol{p}_b$ and $\boldsymbol{p}_0$ determines an expression for λ_b in terms of λ_a. Similarly, we can determine an expression for λ_c in terms of λ_b and, hence, in terms of λ_a. Finally, λ_d may also be expressed in terms of λ_a. However, the planarity of the star with vertices $\boldsymbol{p}_a, \boldsymbol{p}_d, \boldsymbol{p}_{ad}$ and $\boldsymbol{p}_0$ allows us to write λ_a in terms of λ_d, which determines λ_a uniquely. Hence, we have established explicit expressions for $\boldsymbol{p}_a$, $\boldsymbol{p}_b$, $\boldsymbol{p}_c$ and $\boldsymbol{p}_d$ in terms of $\boldsymbol{p}_{ab}$, $\boldsymbol{p}_{ad}$, $\boldsymbol{p}_{cd}$ and $\boldsymbol{p}_{bc}$. In order to verify the $\mathcal{C}^1$ condition between Q_a and Q_b, we now reparametrise the two quadrics Q_a and Q_b according to

$$\begin{aligned} Q_a &\sim \boldsymbol{p}_{ab} + \tilde{\mu}_\alpha \boldsymbol{p}_a + \tilde{\nu}_\alpha \boldsymbol{p}_0 + \tilde{\mu}_\alpha \tilde{\nu}_\alpha \boldsymbol{p}_{ad} \\ Q_b &\sim \boldsymbol{p}_{ab} + \tilde{\mu}_\beta \boldsymbol{p}_b + \tilde{\nu}_\beta \boldsymbol{p}_0 + \tilde{\mu}_\beta \tilde{\nu}_\beta \boldsymbol{p}_{bc}, \end{aligned}$$

where $\tilde{\mu}_\alpha = 1/\mu_\alpha$ and $\tilde{\nu}_\alpha = \nu_\alpha$. Any point X on the common edge between Q_a and Q_b is then parametrised in two ways, namely

$$X \sim \boldsymbol{p}_{ab} + \tilde{\nu}_\alpha \boldsymbol{p}_0 \sim \boldsymbol{p}_{ab} + \tilde{\nu}_\beta \boldsymbol{p}_0$$

for suitable parameters $\tilde{\nu}_\alpha$ and $\tilde{\nu}_\beta$ which are evidently related by $\tilde{\nu}_\alpha = \tilde{\nu}_\beta$. The $\mathcal{C}^1$ condition is then given by

$$\left| \boldsymbol{p_0},\, X,\, \frac{\partial}{\partial \tilde{\mu}_\alpha} Q_a,\, \frac{\partial}{\partial \tilde{\mu}_\beta} Q_b \right| = 0,$$

evaluated at $\tilde{\nu}_\alpha = \tilde{\nu}_\beta$. This reduces to

$$|\boldsymbol{p}_0,\, \boldsymbol{p}_{ab},\, \boldsymbol{p}_a + \tilde{\nu}_\alpha \boldsymbol{p}_{ad},\, \boldsymbol{p}_b + \tilde{\nu}_\alpha \boldsymbol{p}_{bc}| = 0,$$

so that

$$|\boldsymbol{p}_0,\, \boldsymbol{p}_{ab},\, \boldsymbol{p}_{ad},\, \boldsymbol{p}_b| + |\boldsymbol{p}_0,\, \boldsymbol{p}_{ab},\, \boldsymbol{p}_a,\, \boldsymbol{p}_{bc}| = 0,$$

by virtue of the co-planarity of the points $\boldsymbol{p}_0$, $\boldsymbol{p}_{ab}$, $\boldsymbol{p}_{ad}$, $\boldsymbol{p}_{bc}$ and the points $\boldsymbol{p}_0$, $\boldsymbol{p}_{ab}$, $\boldsymbol{p}_a$, $\boldsymbol{p}_b$. The above-mentioned expressions for $\boldsymbol{p}_a$ and $\boldsymbol{p}_b$ may now be used to verify that this condition is identically satisfied.

We now return to the even sublattice of the Demoulin lattice (Figure 5). Then, associated with the quadrilateral $[\boldsymbol{r}, \boldsymbol{r}_{\bar{1}\bar{2}}, \boldsymbol{r}_{\bar{2}\bar{2}}, \boldsymbol{r}_{1\bar{2}}]$ is a quadric $Q_{\bar{2}}$, which is the Lie quadric corresponding to the surface $\Sigma_{\bar{2}}$ at the point $\boldsymbol{r}_{\bar{2}}$. In the same manner, we may associate Lie quadrics Q_1, $Q_{\bar{1}}$ and Q_2 with the quadrilaterals $[\boldsymbol{r}, \boldsymbol{r}_{1\bar{2}}, \boldsymbol{r}_{11}, \boldsymbol{r}_{12}]$, $[\boldsymbol{r}, \boldsymbol{r}_{\bar{1}\bar{2}}, \boldsymbol{r}_{\bar{1}\bar{1}}, \boldsymbol{r}_{\bar{1}2}]$ and $[\boldsymbol{r}, \boldsymbol{r}_{\bar{1}2}, \boldsymbol{r}_{22}, \boldsymbol{r}_{12}]$ respectively of the even sublattice. We can think of $Q_{\bar{2}}$, Q_1, Q_2 and $Q_{\bar{1}}$ as playing the role of Q_a, Q_b, Q_c and Q_d in Theorem 9. Similarly, we may also associate a Lie quadric with each quadrilateral of the odd sublattice. In particular, we may think of the quadric Q passing through the edges of the quadrilateral $[\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_{\bar{1}}, \boldsymbol{r}_{\bar{2}}]$ of the odd sublattice as being the quadric $\hat{Q}$ in Theorem 9. $\hat{Q}$ touches the planar star of the even sublattice at $\boldsymbol{r}$ and, moreover, as a result of the W-congruence property, the quadrilateral $[\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_{\bar{1}}, \boldsymbol{r}_{\bar{2}}]$ touches the quadrics Q_1, Q_2, $Q_{\bar{1}}$ and $Q_{\bar{2}}$ at the points $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\boldsymbol{r}_{\bar{1}}$ and $\boldsymbol{r}_{\bar{2}}$ respectively. Thus, the even sublattice and the quadrics associated with the quadrilaterals of this sublattice satisfy the conditions of Theorem 9. Hence, the quadrics $Q_{\bar{2}}$, Q_1, Q_2 and $Q_{\bar{1}}$ satisfy the $\mathcal{C}^1$ condition. Similarly, the quadrics associated with the quadrilaterals of the odd sublattice satisfy the $\mathcal{C}^1$ condition. Thus, we have come to the following conclusion.

Theorem 10 *Let Σ be the Demoulin lattice of a projective minimal surface evaluated at a point (x, y). Then, the quadrics associated with the quadrilaterals of the even (or odd) sublattice of Σ satisfy the $\mathcal{C}^1$ condition and therefore constitute a set of lattice Lie quadrics for the even (or odd) sublattice.*

3.2 Discrete envelopes and discrete projective minimal surfaces

In order to proceed, it is now required to introduce the notion of discrete envelopes of lattice Lie quadrics.

Definition 5 A discrete envelope of a set of lattice Lie quadrics $\{Q\}$ is a $\mathbb{Z}^2$ lattice such that each star touches the corresponding lattice Lie quadric.

Remark 6 *By construction, a discrete envelope constitutes a discrete asymptotic net since each star lies in the tangent plane of the corresponding lattice Lie quadric at the point of contact. It is evident that the existence of a discrete envelope represents a constraint on the original discrete asymptotic net.*

As a result of Theorem 10, we can now think of the even and odd sublattices of the Demoulin lattice, evaluated at a point (x, y), as discrete envelopes of lattice Lie quadrics since the planar stars of one sublattice touch the lattice Lie quadrics of the other sublattice. In the classical case, we say that a surface Σ and one of its Demoulin transforms are in asymptotic correspondence if the asymptotic coordinates on Σ are mapped to the asymptotic coordinates on the Demoulin transform. We use the following theorem relating a surface in $\mathbb{RP}^3$ and its Demoulin transforms to motivate discretisation [3, 10].

Theorem 11 *A surface Σ and at least one of its Demoulin transforms are in asymptotic correspondence if and only if Σ is projective minimal or a Q-surface.*

Remark 7 *We call a surface which is in asymptotic correspondence with one of its Demoulin transforms a PMQ-surface. Separating Q-surfaces from projective minimal surfaces requires additional geometric and algebraic analysis which is beyond the scope of this paper. We note, however, that at least one of the envelopes of a Lie quadric of a Q-surface is itself a quadric [3].*

In view of Remark 6, the following definition is natural.

Definition 6 A discrete PMQ-surface is a discrete asymptotic net $[\boldsymbol{r}] : \mathbb{Z}^2 \to \mathbb{RP}^3$ which admits an associated discrete envelope.

Remark 8 *By construction, discrete PMQ-surfaces are discrete analogues of projective minimal or Q-surfaces. Even though the above definition essentially captures the definition of discrete projective minimal surfaces by factoring out discrete analogues of Q-surfaces, as in the continuous case, differentiating discrete Q-surfaces and discrete projective minimal surfaces requires further analysis. This is the subject of a separate publication.*

We conclude the paper with the following key result.

Theorem 12 *The even and odd sublattices of the Demoulin lattice of a projective minimal surface, evaluated at a point (x, y), constitute discrete PMQ-surfaces.*

References

1. Bobenko, A., Pinkall, U.: Discrete surfaces with constant negative Gaussian curvature and the Hirota equation. J. Diff. Geom. **43**, 527–611 (1996)
2. Bobenko, A.I., Suris, Y.B.: Discrete differential geometry, *Graduate Studies in Mathematics*, vol. 98. American Mathematical Society, Providence, RI (2008)
3. Bol, G.: Projektive Differentialgeometrie. 2. Teil. Vandenhoeck & Ruprecht, Göttingen (1954)
4. Ferapontov, E.V.: Integrable systems in projective differential geometry. Kyushu J. Math. **54**(1), 183–215 (2000)
5. Hirota, R.: Nonlinear partial difference equations. III. Discrete sine-Gordon equation. J. Phys. Soc. Japan **43**, 2079–2086 (1977)
6. Huhnen-Venedey, E., Rörig, T.: Discretization of asymptotic line parametrizations using hyperboloid surface patches. Geom. Dedicata **168**, 265–289 (2014)
7. Huhnen-Venedey, E., Schief, W.K.: On Weingarten transformations of hyperbolic nets. Int. Math. Res. Notices **2015**, 2021–2081 (2015)
8. Mayer, O.: Contribution á l'étude des surfaces minima projectives. Bull. Sc. Math pp. 146–168,188–200 (1932)
9. Rogers, C., Schief, W.K.: Bäcklund and Darboux transformations. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge (2002)
10. Sasaki, T.: Line congruence and transformation of projective surfaces. Kyushu J. Math. **60**(1), 101–243 (2006)
11. Sauer, R.: Parallelogrammgitter als Modelle pseudosphärischer Flächen. Math. Z. **52**, 611–622 (1950)
12. Su, B.: On Demoulin transforms of projective minimal surfaces. I. Sci. Sinica **6**, 941–965 (1957)
13. Thomsen, G.: Über eine liniengeometrische Behandlungsweise der projektiven Flächentheorie und die projektive Geometrie der Systeme von Flächen zweiter Ordnung. Abh. Math. Sem. Univ. Hamburg **4**(1), 232–266 (1925)
14. Wunderlich, W.: Zur Differenzengeometrie der Flächen konstanter negativer Krümmung. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. II **160**, 39–77 (1951)