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Learning environments that support the development of multiplicative thinking

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LEARNING ENVIRONMENTS THAT SUPPORT THE DEVELOPMENT OF MULTIPLICATIVE THINKING

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Given the right learning environments primary aged children can and do develop the capacity to think multiplicatively. Through vignettes taken from interviews with a Year 5 class during a research project, the optimal conditions for the conceptual development underpinning multiplicative thinking is examined.

There is a nation-wide concern that we are not producing enough students with a mathematics background. Over the last few years there has been a continuing trend of students electing not to choose higher or even intermediate mathematics as a subject of study (Kennedy, Lyons & Quinn, 2014). There are many possible reasons for this. One of them is that many students choose not to be involved in higher mathematics due to a lack of success during primary and lower secondary years. The reasons behind this lack of success are many and varied, but certainly a major contributor would be an inability to cope with the content of the mathematical curriculum. One area of the mathematical curriculum that causes concern and is really a 'gate keeper' to higher mathematics is the capacity to think multiplicatively.

Multiplicative thinking is not just about multiplication facts

Most teachers would agree that basic multiplication facts are essential for further success in mathematics. They sit at the root of learning multi-digit multiplication, fractions, ratios, division, and decimals (Elkins, 2002; Kilpatrick, Swafford & Findell, 2001). Fluency with multiplication facts allows students to focus on more sophisticated tasks such as problem solving (Westwood, 2003). The authors of this paper whole-heartedly endorse the learning of multiplication facts, but students can be fluent with multiplication facts yet still be additive, rather than multiplicative thinkers. What is required is the development of the ability to apply these facts to the variety of situations that are founded on multiplication.

What is multiplicative thinking?

According to Siemon et al. (2006a, p. 28), multiplicative thinking is:

- a capacity to work flexibly and efficiently with an extended range of numbers (and the relationships between them);

- an ability to recognise and solve a range of problems involving multiplication and/or division including direct and indirect proportion; and
- the means to communicate this effectively in a variety of ways (e.g., words, diagrams, symbolic expressions, and written algorithms).

Whilst most teachers would recognise and acknowledge the first two dot points they may be a little more unfamiliar with the notion of communicating these ideas. However if one takes into consideration the Proficiency Strand requirements from the *Australian Curriculum: Mathematics* (ACARA, 2015) it should be noted that communication is an integral element of the Reasoning Proficiency and therefore fundamental to mathematics learning. It is these same communication skills that allow students to explain their understandings so that teachers can make judgments about their students' depth of knowledge for the purposes of assessment.

Why is multiplicative thinking important?

Multiplicative thinking is critical to mathematical success as it underpins the majority of work pursued in the area of Number and Algebra through the upper primary and lower secondary years of schooling. It is fundamental to the development of concepts and understandings such as algebraic reasoning (Brown & Quinn, 2006), place value, proportional reasoning, rates and ratios, measurement, and statistical sampling (Mulligan & Watson, 1998; Siemon, Izard, Breed & Virgona, 2006b).

Multiplicative thinking and primary school students

The development of multiplicative thinking is obviously important and this raises the question of whether it is being appropriately developed in primary schools. The research of Clark and Kamii (1996) revealed that 52% of Year 5 students were not sound multiplicative thinkers. This was supported through research conducted by Siemon, et al. (2006a) who found that up to 40% of Years 7 and 8 students performed below curriculum expectations in multiplicative thinking and at least 25% were well below expected level.

It is important to note the areas where students might find difficulties in developing multiplicative thinking. The following list is quite long, but what should be highlighted is that there are ways and means to remediate these difficulties that are developmentally suitable for primary school students and develop a conceptual rather than procedural understanding.

Some of the difficulties that students encounter are:

- seeing the relevance of the many-to-one count (Jacob & Willis, 2003);
- employing a row by column structure (an array) to work out a number of squares, and resorting to additive strategies (Batista, 1999);
- recognising the multiplicative situation to employ multiplicative thinking (Van Dooren, De Bock & Verschaffel, 2010);
- reconceptualising their early counting and additive understanding about number to understand multiplicative relationships (Sophian & Madrid, 2003; Wright, 2011);
- recognising that multiplicative thinking is distinctly different from additive thinking even though it is constructed by children from their additive thinking processes (Clark & Kamii, 1996);

- understanding the relationship between multiplication and division and being able to consistently employ the inverse relationship between the two operations (Jacob & Willis, 2003);
- understanding and employing the commutative property of multiplication;
- transitioning from using manipulative materials when the need to describe when the operations of multiplication and division became objects of thought rather than actions (Sophian & Madrid, 2003; Wright, 2011).

The remainder of this article will focus on students from a Year Five classroom and the conceptual understandings that were displayed when students responded to a written quiz and a one-to-one interview regarding multiplicative thinking.

One-to-one interviews

A series of one-on-one interviews were undertaken across several schools. Although acknowledging that one-to-one interviews are time consuming, this form of data collection was chosen as a way in which to:

- uncover children's thinking and help to understand why some students learn and others fail to learn (Heng & Sudarshan, 2013);
- enhance the knowledge and skills of teachers by developing a deeper understanding and awareness of the way that children construct their mathematical understandings (Heng & Sudarshan, 2013);
- illuminate student misconceptions (Heng & Sudarshan, 2013) and reveal conceptual understanding of a topic which may not necessarily be obvious through a written assessment task (Clements & Ellerton, 1995); and
- allow teachers to develop more realistic expectations of what the student can and cannot achieve (Clarke, Roche & Mitchell, 2011)

We would like to explore a selection of the comments made by some of the students during the interview, discuss how these comments have the potential to give an insight into the capacity to think multiplicatively, and highlight some of the teaching practices that may be helpful in assisting the students to develop this thinking. For the purposes of this article we will look in some depth at one of the interviews, with supplementary comments taken from observations from other interviews.

James (a pseudonym) approximately 11 years of age

When James answered the questions posed to him in the interview he was not the student who was the quickest to respond. In answering the question of 7×6, he took a considered moment and then answered the question correctly. Later on, when explaining a mental computation strategy, he made a multiplication fact error of $3 \times 8 = 16$, so his recall was not infallible. However, what he was able to do was to analyse his solution and determine that 16 could not have possibly been a correct answer. James may have made a multiplication recall error but his understanding of numbers allowed him to appreciate there was an error, locate that error and then fix it.

James was asked to use mental computation to work out the answer to 6×17 and to articulate how he would solve the problem. He reasoned as follows: "Six times seven is 42, so, then I think what's six times one? And I think six, and add it to 60, so 42 plus 60 is 102. The 60 came from six times one but it's actually a ten." James's explanation allowed us to see a number of important mathematical ideas and understandings in action. In order to achieve this computation mentally, he applied some basic facts (6 × 1 and 6 × 7), partitioning (seeing seventeen along place value partitioning lines as 10 and 7) and place value (the one is seen as a ten). When James was asked to illustrate his thinking using manipulative materials he was able to do so. Selecting from a range of materials James chose the Multibase Attribute Blocks (MAB) and modelled his understanding by creating six lots of seventeen, each made up of a long (ten) and seven ones. It should be noted that he actually talked about needing six tens and then about the required ones, not one ten and seven ones and another one ten and seven ones, and another...etc. His thinking suggested he was seeing the objects multiplicatively rather than additively.

James was then asked if, by knowing 6 × 17, did he know what 17 × 6 was? His answer to this was a very firm "Yes". When asked to explain how he knew, James expressed his understanding by saying it was "the same thing backwards", and then proceeded to explain how he was still multiplying 6 by 10 and 6 by 7, hinting at an understanding of the commutative property of multiplication. He then went on to use smaller numbers to illustrate his understanding. He selected counters to illustrate five multiplied by three and constructed a multiplicative array. That is he arranged the counters as three rows of five counters in a rectangle (Figure 1).

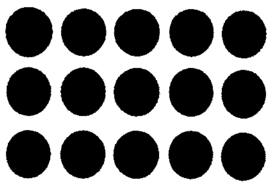

Figure 1. A three by five array.

He then said "You can also look at it as three columns and five rows" and turned the rectangle to illustrate a five by three array (Figure 2).

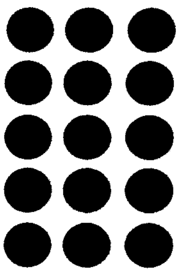

Figure 2. A five by three array.

Again, this level of understanding strongly indicated multiplicative thinking. Other students when faced with constructing a model to illustrate three by five created three sets of five (Figure 3) and when asked if there was another way of showing this did not employ the array, perhaps indicating that additive thinking was dominant.

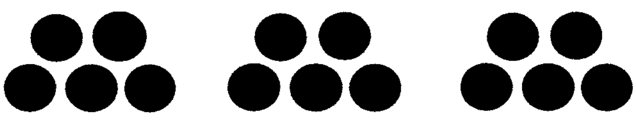

Figure 3. Three sets of five.

When asked if multiplying 89 by three gave the same answer as 80 by three and nine by three James was unequivocal in answering in the affirmative. He stated that splitting it up to "three times 80 and three times nine" was easier than trying to work out 89 times three or 89 and 89 and 89 (James indicated a developing understanding of the limitations of repeated addition/additive thinking). Although he did not name it, nor was he required to, James showed himself to be completely comfortable in partitioning a number and employing the distributive property of multiplication over addition. Likewise when James was asked to explain the inverse relationship between multiplication and division he did so without hesitation.

James was asked the question: "My friend Paul says that if you know that 6 × 17 is 102, you can work out the answer to 102 ÷ 6. Is he correct?" His reply was "that (pointing to 102) is a multiple of six and 17... I know a multiple divided by a factor is another factor." Not only had James displayed an application of the inverse property of multiplication and division, his articulation of it was impressive. It should be worth noting that several students from the same classroom as James produced well-informed and articulate answers and were obviously well used to verbalising their reasoning and understanding.

At a different point of the interview James used the term "add a zero" when he was multiplying by ten. When probed about the notion of "adding a zero" he was able articulate that by adding a zero you were actually "timesing by 10". Initially using the phrase "add a zero" may suggest a naïve understanding of what happens to a number when multiplied by ten. However, when asked what the answer was to 3.6 multiplied by 10, James did not over-generalise this rule, he stated: "I can't add a zero onto the end of this 'cause that just makes the fraction longer." He was then asked if that made the number any different, to which he replied, "No it doesn't make it any different, so I just move the six one space up, past the... um... decimal point... to make it 36." James showed a sound understanding in that he talked about moving the digit one place rather than moving the decimal point, and he was very aware that adding a zero to 3.6 did not alter the magnitude of the number, that is, 3.6 is equal to 3.60. Another observation which can be made was that James related the numbers to the right of the decimal point as being fractions, something which not all students did. Later he expressed that two point six meant two and six-tenths. He further explained that "timesing by ten is basically having a number line, like you'd have the millions, one hundred thousands, ten thousands, down to one and then into the fractions". Although he may have used the term number line a bit loosely, James strongly indicated a knowledge of the place value system.

When the conversation was steered towards division as a multiplicative situation James seemed equally relaxed. He referred to the inverse relationship between multiplication and division and how he employed multiplication when presented with division problems. He also explained that division does not always provide a smaller answer and used the examples of dividing by one and dividing by fractions. Again he talked about moving the number (digit) when dividing by ten and correctly articulated that the digit moves to the left and also mentioned that the resulting number was ten times smaller. When shown the card 8 ÷ 0.1 and asked to predict the answer James immediately answered with zero point eight (0.8). When asked to check this answer using a calculator and seeing that the answer was 80, James responded by talking about being exposed recently in the classroom to some reasoning. He said that "the 80 is a bigger digit but it's actually smaller pieces." He was asked to clarify and after a little hesitation he said, "You'd have how many ones go into eight, so you'd have eight ones. So you'd have eight ones and the ones are the parts. (A prompt: "How many parts?"). There's 10... there would be ten." James has reasoned that 10 'pieces' per whole multiplied by eight gives a total of 80 'pieces'. As he had a conceptual understanding regarding fractions, decimals, place value and multiplication he was able to reason a construct by making connections through and with these understandings.

Teaching experiences

After the student interviews were carried out and analysed, it was decided to spend some time discussing the teaching experiences that James' teacher could identify as contributing to the development of multiplicative thinking in this particular Year Five classroom. A taped semi-structured interview was undertaken and the following points emerged from this interview:

- much time was spent on developing place value concepts and the role zero plays in place value;
- there was a focus on discussions in the mathematics classroom;
- the development of mathematical vocabulary was seen as important;
- development of multiplication facts was seen as a reasoning activity as well as a fluency activity;
- parents were involved in assisting students with their multiplication facts (and were shown how to do this);
- division was introduced before multiplication and making connections between the two was seen as a priority;
- students were given permission to learn at their own developmental stage;
- fractions and decimal fractions were investigated together;
- a lot of time was spent on building mathematical connections; and
- a collaborative culture existed between the teachers in the school and there was a school vision of how to develop mathematical concepts.

As evidenced by the student interviews it was clear that this teacher was successful in developing multiplicative thinking in this Year Five class. One interesting point that arose from the teacher interview was the notion that the use of arrays was highlighted in the Year Four classes in the school and that the Year Five teacher was able to build on the work done in the previous year. This whole school approach was mentioned on several occasions during the teacher interview and was obvious during the student interviews, as the few students who took an entirely algorithmic approach to multiplication were all new to the school.

Conclusion

We are not suggesting that James is necessarily a *typical* Year Five student, but he *is* a Year Five student. It should be noted that to a lesser or greater extent, all of the understandings he articulated were also articulated by his peers. What James and his peers showed the authors was that multiplicative thinking is achievable in the primary school setting. The many mathematical ideas which underpin multiplicative thinking are part of the primary curriculum and sound pedagogical practices can help the students towards developing a conceptual understanding of this vital piece of the mathematics jigsaw.

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